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On Sheffer polynomial families

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Abstract – Attention is focused to particular families of Sheffer polynomials which are different from the classical ones because they satisfy non-standard differential equations, including some of fractional type. In particular Sheffer polynomial families are considered whose characteristic elements are based on powers or exponential functions.

Keywords: Sheffer polynomial families, Generating functions, Monomiality principle, Fractional derivative equations, Combinatorial analysis

1 Introduction

In recent articles [1, 2], new sets of Sheffer [3] and Brenke [4] polynomials, based on higher order Bell numbers [5–9], have been studied. Furthermore, several integer sequences [10] associated with the considered polynomials sets both of exponential [11, 12] and logarithmic type have been introduced [2].

We recall that the exponential and logarithmic polynomials have been recently studied even in the multidimensional case [13–15].

It is worth to note that the Sheffer family includes a plenty of unusual polynomials, which satisfy non-standard differential equations. In this article we focus our attention on Sheffer polynomial families whose characteristic elements are based on powers or exponential functions, deriving the relevant differential equations, which are frequently of fractional type.

2 Sheffer polynomials

We start recalling the particular meaning of the term *set* in the framework of polynomial theory.

Definition 2.1. *A polynomial family* $\{P_n(x)\}_{(n\geq 0)}$ *is called a polynomial set iff* $\forall n$, $\deg P_n = n$.
In what follows, we are dealing with polynomial families that, in several cases, don't satisfy the above condition.
The Sheffer polynomial families $\{s_n(x)\}$ are introduced [3] by means of the exponential generating function [16] of the type:

$$A(t)\exp(xH(t)) = \sum_{n=0}^{\infty} s_n(x)\,\frac{t^n}{n!}, \tag{1}$$

where,

$$\begin{aligned} A(t) &= \sum_{n=0}^{\infty} a_n\,\frac{t^n}{n!}, \quad (a_0\neq 0),\\ H(t) &= \sum_{n=0}^{\infty} h_n\,\frac{t^n}{n!}, \quad (h_0 = 0)\,. \end{aligned} \tag{2}$$

Remark 2.2. It is well known [4, 17] that there is a natural link between the function $H(t)$ and the degree of polynomials $s_n(x)$ in expansion (1). Namely,

$$\deg s_n = n\;iff, \quad \text{in equation} \quad (2), \quad h_1\neq 0.$$

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Actually, in what follows, *if* $H(t)$ *is a polynomial of degree* m, we have found that $\deg s_n \leq [\frac{n}{m}]$, where $[\cdot]$ denotes the integral part.
In general, we are dealing with a Sheffer polynomial set *iff* the condition $h_1 \neq 0$ is satisfied.
According to a different characterization (see [18], p. 18), the same polynomial sequence can be defined by means of the pair $(g(t), f(t))$, where $g(t)$ is an invertible series and $f(t)$ is a delta series:

$$\begin{aligned} g(t) &= \sum_{n=0}^{\infty} g_n \frac{t^n}{n!}, \quad (g_0 \neq 0), \\ f(t) &= \sum_{n=0}^{\infty} f_n \frac{t^n}{n!}, \quad (f_0 = 0, f_1 \neq 0). \end{aligned} \tag{3}$$

Denoting by $f^{-1}(t)$ the compositional inverse of $f(t)$ (i.e., such that $f(f^{-1})(t)) = (f^{-1})(f(t)) = t$), the exponential generating function of the sequence $\{s_n(x)\}$ is given by

$$\frac{1}{g[f^{-1}(t)]} \exp\left(xf^{-1}(t)\right) = \sum_{n=0}^{\infty} s_n(x) \frac{t^n}{n!}, \tag{4}$$

so that,

$$A(t) = \frac{1}{g[f^{-1}(t)]}, \quad H(t) = f^{-1}(t). \tag{5}$$

When $g(t) \equiv 1$, the Sheffer sequence corresponding to the pair $(1, f(t))$ is called the associated Sheffer sequence $\{\sigma_n(x)\}$ for $f(t)$, and its exponential generating function is given by

$$\exp\left(xf^{-1}(t)\right) = \sum_{n=0}^{\infty} \sigma_n(x) \frac{t^n}{n!}. \tag{6}$$

A list of known Sheffer polynomial sequences and their associated ones can be found in [19]. New Euler-type Sheffer polynomials have been recently introduced in [20].

3 Power based Sheffer polynomials

In this section we derive Sheffer polynomial families assuming the following basic functions:

$$\frac{A'(t)}{A(t)} = t^p, \quad H(t) = t^q, \tag{7}$$

p and q positive integer numbers, so that,

$$A(t) = \exp\left(\frac{t^{p+1}}{p+1}\right), \tag{8}$$

and the generating function is:

$$G(t,x) = \exp\left(\frac{t^{p+1}}{p+1} + xt^q\right) = \sum_{n=0}^{\infty} \tilde{q}_n(p;q;x) \frac{t^n}{n!}. \tag{9}$$

Therefore, denoting by $H^{-1}(t)$ the compositional inverse of $H(t)$, we have:

$$H^{-1}(t) = t^{1/q}, \quad H'(t) = q\, t^{q-1}. \tag{10}$$

Definition 3.1. We recall that a polynomial set $\{p_n(x)\}$ is called quasi-monomial if and only if there exist two operators $\hat{P}$ and $\hat{M}$ such that,

$$\begin{aligned} \hat{P}(p_n(x)) &= np_{n-1}(x), \\ \hat{M}(p_n(x)) &= p_{n+1}(x), \quad (n = 1, 2, \ldots). \end{aligned} \tag{11}$$

$\hat{P}$ is called the *derivative* operator and $\hat{M}$ the *multiplication* operator, as they act in the same way of classical operators on monomials.

This definition traces back to a paper by J.F. Steffensen [21], recently improved by G. Dattoli [22] and widely used in several applications (see e.g., [23, 24] and the references therein).
Y. Ben Cheikh [25] proved that every polynomial set is quasi-monomial under the action of suitable derivative and multiplication operators. In particular, if the considered polynomial set is Sheffer, the Corollary 3.2 in the same article ensure that the derivative and multiplication operator are given by:

$$\begin{aligned} \hat{P} &= H^{-1}(D_x), \\ \hat{M} &= \frac{A'[H^{-1}(D_x)]}{A[H^{-1}(D_x)]} + xH'[H^{-1}(D_x)], \end{aligned} \tag{12}$$

where prime denotes the derivative with respect to t, and D_x the derivative with respect to x.

Remark 3.2. It is worth to note that the above mentioned result (Corollary 3.2 in [25]), given for polynomial sets, never uses in proof the condition $h_1 \neq 0$. Therefore, it can be applied even to polynomials defined by Sheffer generating functions (1), i.e. to Sheffer polynomial families.

According to the above equations (8), (10), (12), we have the result:

Theorem 3.3. *The derivative and multiplication operators of the Sheffer polynomial family defined by the generating function (9) are given by,*

$$\begin{aligned} \hat{P} &= D_x^{1/q}, \\ \hat{M} &= D_x^{p/q} + q\,x\,D_x^{(q-1)/q}\,. \end{aligned} \tag{13}$$

3.1 Differential equation

As a consequence of the monomiality principle, the factorization method gives the differential equation satisfied by the quasi-monomial polynomials $\{p_n(x)\}$ in the form:

$$\hat{M}\hat{P}\,p_n(x) = n\,p_n(x)\,. \tag{14}$$

In the present case, we have the result:

Theorem 3.5. *The Sheffer polynomials $\{\tilde{q}_n(p,q;x)\}$ satisfy the differential equation*

$$\left(D_x^{(p+1)/q} + q\,x\,D_x\right)\tilde{q}_n(p,q;x) = n\,\tilde{q}_n(p,q;x)\,. \tag{15}$$

4 Some particular examples

We show in this Section some particular example.

4.1 Case $p = 3, q = 2, \quad G(t,x) = \exp[t^{\frac{4}{4}} + xt^2]$

The ordinary differential equation is:

$$\left(D_x^2 + 2\,x\,D_x\right)\tilde{q}_n(3,2;x) = n\,\tilde{q}_n(3,2;x)\,. \tag{16}$$

The first few $\tilde{q}_n(3,2;x)$ polynomials are as follows:

$$\begin{aligned} &\tilde{q}_0(3,2;x) = 1, \\ &\tilde{q}_{2k+1}(3,2;x) = 0, \quad \forall k \geq 0, \\ &\tilde{q}_2(3,2;x) = 2x, \\ &\tilde{q}_4(3,2;x) = 6(2x^2+1), \\ &\tilde{q}_6(3,2;x) = 180(4x^3+x), \\ &\tilde{q}_8(3,2;x) = 420(4x^4+12x^2+3)\,. \end{aligned}$$

Further values can be easily achieved by using Wolfram Alpha©.

Remark 4.1. Note that there is a link of the $\tilde{q}_n(3,2;x)$ polynomials with the classical $H_n(x)$. In fact, let

$$G_1(x,t) = \exp(2x\tau - \tau^2) = \sum_{n=0}^{\infty} H_n(\tau)\,\frac{\tau^n}{n!}$$

$$G_2(x,t) = \exp(x\,t^2 + t^4/4) = \sum_{n=0}^{\infty} \tilde{q}_n(3,2;x)\,\frac{t^n}{n!}\,.$$

Since $G_2(x,-t) = G_2(x,t)$, it follows that $q_{2k+1}(3,2;x) = 0$, $\forall k \geq 0$, and

$$G_2(x,t) = \sum_{n=0}^{\infty} \tilde{q}_{2n}(3,2;x)\,\frac{t^{2n}}{2n!}\,.$$

Therefore, putting $t^2 = 2\mathrm{i}s$ (i the imaginary unit), we find,

$$G_1(\mathrm{i}x,s) = \exp(2\mathrm{i}xs - \mathrm{s}^2)\sum_{n=0}^{\infty}(2\,\mathrm{i})^n\,\tilde{q}_{2n}(3,2;x)\,\frac{s^n}{2n!},$$

and consequently,

$$\tilde{q}_{2n}(3,2;x) = \frac{(2n)!}{2^n\,n!}\,\frac{H_n(\mathrm{i}x)}{\mathrm{i}^n} = (2n-1)!!\,\frac{H_n(\mathrm{i}x)}{\mathrm{i}^n}\,.$$

In a similar way, when $\frac{p+1}{q} = N$, (N integer number), we can find a link between the Gould-Hopper polynomials defined by the generating function,

$$G_1^*(x,t) = \exp\left(x\tau + \tau^N\right) = \sum_{n=0}^{\infty} H_n^{(N)}(\tau)\,\frac{\tau^n}{n!},$$

and the Sheffer polynomials defined by,

$$G_2^*(x,t) = \exp\left(x\,t + t^{(p+1)/q}\right) = \sum_{n=0}^{\infty} \tilde{q}_n(p,q;x)\,\frac{t^n}{n!}.$$

In fact, owing the symmetry of these polynomials, we have,

$$\tilde{q}_{Nn+k}(p,q;x) = 0, \quad \forall n \geq 0,\ k = 1,2,\ldots,N-1,$$

and therefore, putting $\tau - t^N$, we find:

$$G_2^*(x,t) = \sum_{n=0}^{\infty} \tilde{q}_{Nn}(p,q;x)\,\frac{t^{Nn}}{(Nn)!} = \sum_{n=0}^{\infty} \tilde{q}_n(p,q;x)\,\frac{\tau^n}{(Nn)!}\,,$$

so that $\tilde{q}_n(p,q;x)$, as a polynomial of degree n, can be expressed in terms of the Gould-Hopper polynomials by:

$$\tilde{q}_{Nn}(p,q;x) = \frac{(Nn)!}{n!}\,H_n^{(N)}(x).$$

Remark 4.2.
1. The equation (16) is an ordinary differential equation of order 2, because $(p+1)/q = 4/2 = 2$. This happens, in general, if and only if $\frac{p+1}{q} = N$, (N integer number).
2. Note that, owing to the fractional derivative operator $\hat{P} = D_x^{1/2}$, the polynomial degree increases slowly, actually the degree of $\tilde{q}_{2n}(p,q;x)$ is equal to $(2n)/2 = n$.

4.2 Case $p = 1, q = 3$, $G(t,x) = \exp[\frac{t^2}{2} + xt^3]$

The fractional differential equation is:

$$\left(D_x^{2/3} + 3\,x\,D_x\right)\tilde{q}_n(1,2;x) = n\,\tilde{q}_n(1,3;x)\,. \tag{17}$$

The first few $\tilde{q}_n(1,3;x)$ polynomials are as follows:
If $n \equiv 0$, (mod 3):

$$\begin{aligned}
&\tilde{q}_0(1,3;x) = 1,\\
&\tilde{q}_3(1,3;x) = 6x,\\
&\tilde{q}_6(1,3;x) = 15(24x^2+1),\\
&\tilde{q}_9(1,3;x) = 7560(8x^3+x),\\
&\tilde{q}_{12}(1,3;x) = 10395(1920x^4+480x^2+1).
\end{aligned}$$

If $n \equiv 1$, (mod 3):

$$\begin{aligned}
&\tilde{q}_1(1,3;x) = 0,\\
&\tilde{q}_4(1,3;x) = 3,\\
&\tilde{q}_7(1,3;x) = 630x,\\
&\tilde{q}_{10}(1,3;x) = 945(240x^2+1),\\
&\tilde{q}_{13}(1,3;x) = 1621620(80x^3+x).
\end{aligned}$$

If $n \equiv 2$, (mod 3):

$$\begin{aligned}
&\tilde{q}_2(1,3;x) = 1,\\
&\tilde{q}_5(1,3;x) = 60x,\\
&\tilde{q}_8(1,3;x) = 105(96x^2+1),\\
&\tilde{q}_{11}(1,3;x) = 103950(32x^3+x),\\
&\tilde{q}_{14}(1,3;x) = 135135(13440x^4+840x^2+1).
\end{aligned}$$

Further values can be easily achieved by using Wolfram Alpha©.

Remark 4.3.
1. The equation (17) is a fractional differential equation.
2. Note that, owing to the fractional derivative operator $\hat{P} = D_x^{1/3}$, the degree of the polynomial $\tilde{q}_n(x;3)$ is equal to:

- $n/3$ if $n \equiv 0$, (mod 3),
- $(n-4)/3$ if $n \equiv 1$, (mod 3), and $n \geq 4$,
- $(n-2)/3$ if $n \equiv 2$, (mod 3), and $n \geq 2$.

5 A particular family of Sheffer polynomials

In this Section we consider a particular family of Sheffer polynomials defined by generating functions of the type:

$$G(t,x) = A(t)\exp[xH(t)], \quad \text{where} \quad \frac{A'(t)}{A(t)} = H(t), \tag{18}$$

and $H(t)$ is an invertible function. Therefore, we find,

$$\begin{aligned}
&A(t) = \exp[\textstyle\int H(t)dt], \quad \text{where}\\
&G(t,x) = \exp[\textstyle\int H(t)dt + xH(t)] = \displaystyle\sum_{n=0}^{\infty} \tau_n(x)\,\frac{t^n}{n!}.
\end{aligned} \tag{19}$$

Putting, as before, $f(t) = H^{-1}(t)$, according to the recalled result by Y. Ben Cheikh [25], the derivative and multiplication operators for the relevant polynomials τ_n are given by:

$$\begin{aligned}
&\hat{P} = H^{-1}(D_x),\\
&\hat{M} = D_x + xH'[H^{-1}(D_x)],
\end{aligned} \tag{20}$$

so that we have the result:

Theorem 5.1. *The Sheffer polynomials $\{\tau_n(x)\}$ satisfy the differential equation*

$$[D_x + x\,H'(H^{-1}(D_x))]H^{-1}(D_x)\,\tau_n(x) = n\,\tau_n(x)\,. \tag{21}$$

6 Particular examples

Note that the particular Sheffer polynomials of Section 5 only depend on $H(t)$.

6.1 A power based example, linked to Section 3

We assume,

$$H(t) = \frac{t^{2q+1}}{2q+1}, \tag{22}$$

(q a positive integer number), so that,

$$H'(t) = t^{2q}, \quad H^{-1}(t) = [(2q+1)\,t]^{1/(2q+1)},$$

$$G(t,x) = \exp\left[\frac{1}{2q+1}\left(\frac{1}{2q+2}\,t^{2q+2} + x\,t^{2q+1}\right)\right] = \sum_{n=0}^{\infty} \tilde{\tau}_n(x;q)\,\frac{t^n}{n!}\,. \tag{23}$$

According to the above results, the derivative and multiplication operators for the quasi-monomials $\tilde{\tau}_n(x;q)$ are given by,

$$\hat{P} = [(2q+1)D_x]^{1/(2q+1)},$$

$$\hat{M} = D_x + x\,[(2q+1)D_x]^{2q/(2q+1)}, \tag{24}$$

and the relevant differential equation writes,

$$\left[D_x + x\,[(2q+1)D_x]^{2q/(2q+1)}\right][(2q+1)D_x]^{1/(2q+1)}\tilde{\tau}_n(x;q) = n\,\tilde{\tau}_n(x;q),$$

that is,

$$(2q+1)\,\left[D_x^{(2q+2)/(2q+1)} + xD_x\right]\tilde{\tau}_n(x;q) = n\,\tilde{\tau}_n(x;q), \tag{25}$$

which is a fractional derivative equation.

6.2 Case $q = 1$, $H(t) = t^3/3$

The generating function is,

$$G(t,x) = \exp\left[\frac{1}{3}\left(\frac{1}{4}\,t^4 + x\,t^3\right)\right] = \sum_{n=0}^{\infty} \tilde{\tau}_n(x;3)\,\frac{t^n}{n!},$$

and the fractional differential equation (25) writes,

$$3\,\left[D_x^{4/3} + xD_x\right]\tilde{\tau}_n(x;3) = n\,\tilde{\tau}_n(x;3)\,. \tag{26}$$

The first few $\tilde{\tau}_n(x;3)$ polynomials are as follows:

$$\begin{array}{lll}
\tilde{\tau}_0(x;3) = 1, & \tilde{\tau}_1(x;3) = 0, & \tilde{\tau}_2(x;3) = 0, \\
\tilde{\tau}_3(x;3) = 2\,x, & \tilde{\tau}_4(x;3) = 2, & \tilde{\tau}_5(x;3) = 0, \\
\tilde{\tau}_6(x;3) = 40\,x^2, & \tilde{\tau}_7(x;3) = 140\,x, & \tilde{\tau}_8(x;3) = 140, \\
\tilde{\tau}_9(x;3) = 2240\,x^3, & \tilde{\tau}_{10}(x;3) = 16800\,x^2, & \tilde{\tau}_{11}(x;3) = 46200\,x, \\
\tilde{\tau}_{12}(x;3) = 15400(16\,x^4+3), & \tilde{\tau}_{13}(x;3) = 3203200\,x^3, & \tilde{\tau}_{14}(x;3) = 16816800\,x^2.
\end{array}$$

Further values can be easily achieved by using Wolfram Alpha©.

Note the symmetry of the above scheme, according to which the degree of the polynomial $\tilde{\tau}_n(x;3)$ is equal to:

- $n/3$ if $n \equiv 0$, (mod 3),
- $(n-4)/3$ if $n \equiv 1$, (mod 3), and $n \geq 4$,
- $(n-8)/3$ if $n \equiv 2$, (mod 3), and $n \geq 8$.

6.3 An exponential based example

We assume,

$$H(t) = e^t - 1, \tag{27}$$

so that,

$$H'(t) = e^t, \quad H^{-1}(t) = \log(t+1),$$
$$G(t,x) = \exp\left[e^t - t + x(e^t - 1)\right] = \sum_{n=0}^{\infty} \tilde{\sigma}_n(x) \frac{t^n}{n!} \,. \tag{28}$$

According to the above results, the derivative and multiplication operators for the quasi-monomials $\tilde{\sigma}_n(x)$ are given by,

$$\hat{P} = \log(D_x + 1),$$
$$\hat{M} = D_x + x\,(D_x + 1), \tag{29}$$

and the relevant differential equation writes,

$$[D_x + x\,(D_x + 1)] \log(D_x + 1)\, \tilde{\sigma}(x) = n\, \tilde{\sigma}(x),$$

that is,

$$[D_x + x\,(D_x + 1)] \sum_{k=1}^{n} (-1)^{k+1} \frac{D_x^k}{k} \tilde{\sigma}_n(x) = n\, \tilde{\sigma}_n(x), \tag{30}$$

which is an infinite order differential equation reducing to an equation of order n when applied to a polynomial of degree n.

The first few $\tilde{\sigma}_n(x)$ polynomials are as follows:

$$\begin{aligned}
&\tilde{\sigma}_0(x) = 1,\\
&\tilde{\sigma}_1(x) = x,\\
&\tilde{\sigma}_2(x) = x^2 + x + 1,\\
&\tilde{\sigma}_3(x) = x^3 + 3x^2 + 4x + 1,\\
&\tilde{\sigma}_4(x) = x^4 + 6x^3 + 13x^2 + 11x + 4,\\
&\tilde{\sigma}_5(x) = x^5 + 10x^4 + 35x^3 + 55x^2 + 41x + 11,\\
&\tilde{\sigma}_6(x) = x^6 + 15x^5 + 80x^4 + 200x^2 + 256x^2 + 162x + 41,\\
&\tilde{\sigma}_7(x) = x^7 + 21x^6 + 161x^5 + 595x^4 + 1176x^3 + 1274x^2 + 715x + 162,\\
&\tilde{\sigma}_8(x) = x^8 + 28x^7 + 294x^6 + 1526x^5 + 4361x^4 + 7182x^3 + 6791x^2 + 3425x + 715\,.
\end{aligned}$$

Further values can be easily achieved by using Wolfram Alpha©.

Remark 6.1. Note that the sequence:

$$1, 1, 3, 9, 35, 153, 755, 4105, 24323, \ldots,$$

(that is the values $\tilde{\sigma}_n(1)$), has a combinatorial character, since it appears in the *Encyclopedia of integer sequences* under A217924 – Row sequence of table A217537, $a(n) := \sum_{j=0}^{k}\left(\sum_{k=0}^{n} n_{k-j} 2^j (-1)^{(k-j)}\, \mathrm{Stirling}_2(n-k+j, j)\right)$; – Vladimir Kruchinin, Feb 28, 2015.
Furthermore, the sequence:

$$1, 1, 4, 11, 41, 162, 715, \ldots,$$

(that is the values $\tilde{\sigma}_n(0)$, $n \geq 2$) appears in the *Encyclopedia of integer sequences* under A000296 – Set partitions without singletons: number of partitions of an n-set into blocks of size >1. Also number of cyclically spaced (or feasible) partitions.

7 A mixed-type (power-exp) Sheffer polynomial family

We assume in this Section:

$$H(t) = t^q, \quad A(t) = \exp(e^t - 1), \tag{31}$$

(q a positive integer number), so that,

$$H'(t) = q\, t^{q-1}, \quad H^{-1}(t) = t^{1/q},$$
$$G(t,x) = \exp\left[e^t - 1 + x t^q\right] = \sum_{n=0}^{\infty} \tilde{\rho}_n(x;q) \frac{t^n}{n!} \,. \tag{32}$$

According to the above results, the derivative and multiplication operators for the quasi-monomials $\tilde{\rho}_n(x;q)$ are given by,

$$\begin{aligned} \hat{P} &= D_x^{1/q}, \\ \hat{M} &= \exp\left(D_x^{1/q}\right) + q\,x\,D_x^{(q-1)/q}, \end{aligned} \tag{33}$$

and the relevant differential equation writes,

$$\left[\exp\left(D_x^{1/q}\right)D_x^{1/q} + q\,x\,D_x\right]\tilde{\rho}_n(x;q) = n\,\tilde{\rho}_n(x;q),$$

that is,

$$\left[\sum_{k=0}^{qn-1}\frac{D_x^{(k+1)/q}}{k!} + q\,x\,D_x\right]\tilde{\rho}_n(x;q) = n\,\tilde{\rho}_n(x;q), \tag{34}$$

which is an infinite order differential equation reducing to an equation of order $qn{-}1$ when it is applied to a polynomial of degree n.

7.1 Case $q = 2$, $\mathrm{G}(t, x) = \exp[e^t{-}1 + xt^2]$

The first few $\tilde{\rho}_n(x;2)$ polynomials are as follows:

$$\begin{aligned} \tilde{\rho}_0(x;2) &= 1, \\ \tilde{\rho}_1(x;2) &= 1, \\ \tilde{\rho}_2(x;2) &= 2x + 2, \\ \tilde{\rho}_3(x;2) &= 6x + 5, \\ \tilde{\rho}_4(x;2) &= 12x^2 + 24x + 15, \\ \tilde{\rho}_5(x;2) &= 60x^2 + 50x + 52, \\ \tilde{\rho}_6(x;2) &= 120x^3 + 360x^3 + 450x + 203, \\ \tilde{\rho}_7(x;2) &= 840x^3 + 2100x^2 + 2184x + 877, \\ \tilde{\rho}_8(x;2) &= 1680x^4 + 6720x^3 + 2520x^3 + 11368x + 4140\,. \end{aligned}$$

Further values can be easily achieved by using Wolfram Alpha©.

Remark 7.1.
1. Note that owing to the fractional derivative operator $\hat{P} = D_x^{1/2}$, the degree of the polynomial $\tilde{\rho}_n(x;2)$ is equal to $\left[\frac{n}{2}\right]$.

A similar phenomenon also applies to $q > 2$.
2. Note that the sequence:

$$1, 1, 2, 5, 15, 52, 203, 877, 4140, \ldots,$$

(that is the values $\tilde{\rho}_n(0;2)$), has a combinatorial character, since it appears in the *Encyclopedia of integer sequences* under A000110 – Bell or exponential numbers: number of ways to partition a set of n labeled elements.

8 Conclusion

We have introduced "unusual families" of Sheffer polynomials, namely they satisfy non-standard differential equations, including some of fractional type.

In some case, we have noticed connections with particular integer sequences, since the polynomial – in suitable points – exhibit a combinatorial character. In this article have considered Sheffer polynomials based on powers or exponential functions. In subsequent papers further families will be introduced, by using different basic elements.

Author contributions

The authors claim to have contributed equally and significantly in this paper. Both authors read and approved the final manuscript.

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Compliance with ethical standards

Conflict of interest

The authors declare that they have not received funds from any institution and that they have no conflict of interest.

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