4open 2019, **2**, 17
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https://doi.org/10.1051/fopen/2019009

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www.4open-sciences.org

Special Issue: Difference & Differential Equations and Applications
Guest Editor: Sandra Pinelas

RESEARCH ARTICLE OPEN ACCESS

To the theory of discrete boundary value problems

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Received 7 January 2019, Accepted 5 April 2019

Abstract – We consider discrete analogues of pseudo-differential operators and related discrete equations and boundary value problems. Existence and uniqueness results for special elliptic discrete boundary value problem and comparison for discrete and continuous solutions are given for certain smooth data in discrete Sobolev–Slobodetskii spaces.

Keywords: Digital pseudo-differential operator, Discrete boundary value problem, Periodic factorization, Discrete solution, Error estimate

AMS subject classification: Primary: 35S15, Secondary: 65T50

1 Introduction

As a rule the classical pseudo-differential operator in Euclidean space $\mathbb{R}^m$ is defined by the formula [1, 2]:

$$(Au)(x) = \int_{\mathbb{R}^m} \tilde{A}(x,\xi)e^{-ix\cdot\xi}\tilde{u}(\xi)\mathrm{d}\xi,$$

where the sign $\sim$ over a function denotes its Fourier transform,

$$\tilde{u}(\xi) = \int_{\mathbb{R}^m} u(x)e^{ix\cdot\xi}\mathrm{d}x,$$

and the function $\tilde{A}(x,\xi)$ is called a symbol of a pseudo-differential operator A.

Our main goal here is describing a periodic variant of this definition and studying its certain properties related to solvability of corresponding equations in canonical domains of an Euclidean space. In this paper the main result is related to a comparison of discrete and continuous solutions. We try to preserve maximal correspondence for discrete and continuous cases under digitization, it permits to find more appropriate constructions.

This problem is very large and in our opinion it should include the following aspects according to a lot of physical and technical applications of such operators and related equations:

- finite and infinite discrete Fourier transform as a natural technique for such equations;
- choice of appropriate discrete functional spaces;
- studying solvability for infinite discrete equations;
- studying solvability of approximating finite discrete equations;
- a comparison between continuous and infinite discrete equations;
- a comparison between infinite discrete and finite discrete equations.

This is not completed list of questions for studying which we intend to consider. Some results in this direction were obtained for simplest pseudo-differential operators (Calderon–Zygmund operators [3, 4]) and corresponding equations. Also certain results are related to approximate solutions.

There are few variants of the theory of discrete boundary value problems (see, for example [5, 6]), but these theories are related especially to partial differential operators and do not use the harmonic analysis technique. Since the classical theory

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of pseudo-differential operators is based on the Fourier transform we will use the discrete Fourier transform and discrete analogue of pseudo-differential operators which will include discrete analogues of partial differential and some integral convolution operators.

2 Discrete spaces and digital operators

2.1 Discrete Sobolev–Slobodetskii spaces

Given function u_d of a discrete variable $\tilde{x} \in h\mathbb{Z}^m$, $h > 0$, we define its discrete Fourier transform by the series:

$$(F_d u_d)(\xi) \equiv \tilde{u}_d(\xi) = \sum_{\tilde{x} \in \mathbb{Z}^m} e^{i\tilde{x}\cdot\xi} u_d(\tilde{x}), \xi \in \hbar\mathbb{T}^m,$$

where $\mathbb{T}^m = [-\pi, \pi]^m$, $\hbar = h^{-1}$, and partial sums are taken over cubes,

$$Q_N = \left\{ \tilde{x} \in h\mathbb{Z}^m : \tilde{x} = (\tilde{x}_1, \cdots, \tilde{x}_m), \quad \max_{1 \le k \le m} |\tilde{x}_k| \le N \right\}.$$

We will remind here some definitions of functional spaces [7] and will consider discrete analogue of the Schwartz space $S(h\mathbb{Z}^m)$. Let us denote $\zeta^2 = h^{-2} \sum_{k=1}^{m} (e^{-ih\cdot\xi_k} - 1)^2$ and introduce the following.

Definition 1. *The space $H^s(hZ^m)$ is a closure of the space $S(hZ^m)$ with respect to the norm,*

$$||u_d||_s = \left(\int_{\hbar\mathbb{T}^m} (1 + |\zeta^2|)^s |\tilde{u}_d(\xi)|^2 \mathrm{d}\xi \right)^{\frac{1}{2}}. \tag{1}$$

Fourier image of the space $H^s(h\mathbb{Z}^m)$ will be denoted by $\tilde{H}^s(\hbar\mathbb{T}^m)$.

2.2 Digital pseudo-differential operators

One can define some discrete operators for such functions u_d.

If $\tilde{A}_d(\xi)$ is a periodic function in $\mathbb{R}^m$ with the basic cube of periods $\hbar\mathbb{T}^m$ then we consider it as a symbol. We will introduce a digital pseudo-differential operator in the following way.

Definition 2. *A digital pseudo-differential operator A_d in a discrete domain D_d is called the operator* [7],

$$(A_d u_d)(\tilde{x}) = \sum_{\tilde{y} \in h\mathbb{Z}^m} \int_{\hbar\mathbb{T}^m} \tilde{A}_d(\xi) e^{i(\tilde{x}-\tilde{y})\cdot\xi} \tilde{u}_d(\xi) \mathrm{d}\xi, \tilde{x} \in D_d.$$

We use the class $E_\alpha, \alpha \in \mathbb{R}$, [7] with the following condition:

$$c_1(1 + |\zeta^2|)^{\frac{\alpha}{2}} \le |A_d(\xi)| \le c_2(1 + |\zeta^2|)^{\frac{\alpha}{2}}, \tag{2}$$

and universal positive constants c_1, c_2.

Let $D \subset \mathbb{R}^m$ be a domain. We will study the equation:

$$(A_d u_d)(\tilde{x}) = v_d(\tilde{x}), \quad \tilde{x} \in D_d, \tag{3}$$

in the discrete domain $D_d \equiv D \cap h\mathbb{Z}^m$ and will seek a solution $u_d \in H^s(D_d)$, $v_d \in H_0^{s-\alpha}(D_d)$ [7–9].

Earlier some canonical domains [8–16] were considered but in this paper we will discuss the cases $\mathbb{R}^m, \mathbb{R}^m_+$.

3 Solvability and digital-periodic projectors

3.1 Periodic factorization

This case $\mathbb{R}^m_+$ is very different from $\mathbb{R}^m$, and an ellipticity condition is not sufficient for a solvability. A principal role for the solvability takes a concept of the periodic factorization which is defined for an elliptic symbol.

To describe a solvability picture for equation (3) we introduce the following notations. Let us denote $\Pi_\pm = \{(\xi', \xi_m \pm i\tau), \xi_m \pm i\tau \in \mathbb{C}, \tau > 0\}$, $\xi = (\xi', \xi_m) \in \mathbb{T}^m$.

We will use a special periodic factorization of an elliptic symbol $A_d(\xi) \in E_\alpha$:

$$A_d(\xi) = A_{d,\quad +}(\xi)A_{d,\quad -}(\xi),$$

where the factors $A_{d,\pm}(\xi)$ have some analytical properties in half-strips $\hbar\Pi_\pm$ and satisfy certain estimates [7, 8].

The special index æ of periodic factorization determines the solvability for equation (3), and for special cases we will describe obtained results [7, 8]. These cases are distinct. So, if $|æ - s| < 1/2$ then we have the unique solution:

$$\tilde{u}_d(\xi) = \tilde{A}^{-1}_{d,+}(\xi)P^{\mathrm{per}}_{\xi'}(\tilde{A}^{-1}_{d,-}(\xi)\widetilde{\ell v_d}(\xi)), \tag{4}$$

$$\left(P^{\mathrm{per}}_{\xi'}\tilde{u}_d\right)(\xi) \equiv \frac{1}{2}\left(\tilde{u}_d(\xi) + \frac{h}{2\pi i}v.p.\int\limits_{-\hbar\pi}^{\hbar\pi}\tilde{u}_d\left(\xi', \eta_m\right)\cot\frac{h(\xi_m - \eta_m)}{2}\mathrm{d}\eta_m\right),$$

for equation (3). But if $æ - s = n + \delta, n \in \mathbb{N}, |\delta| < 1/2$ then there are a lot of solutions,

$$\tilde{u}_d(\xi) = \tilde{A}^{-1}_{d,+}(\xi)X_n(\xi)P^{\mathrm{per}}_{\xi'}(X^{-1}_n(\xi)\tilde{A}^{-1}_{d,-}(\xi)\widetilde{\ell v_d}(\xi)) + \tilde{A}^{-1}_{d,+}(\xi)\sum_{k=0}^{n-1}\tilde{c}_k(\xi')\hat{\zeta}^k_m,$$

where $X_n(\xi)$ is an arbitrary polynomial of order n of variables $\hat{\zeta}_k = \hbar(e^{-ih\xi_k} - 1), k = 1, \ldots, m$, satisfying the condition (2), $\tilde{c}_k(\xi'), j = 0, 1, \ldots, n-1$, are arbitrary functions from $\tilde{H}^{s_k}(\hbar\mathbb{T}^{m-1})$, $s_k = s - æ + k - 1/2$.

3.2 Approximation schemes

We will consider the pseudo-differential equation:

$$(Au)(x) = v(x), \quad x \in D, \tag{5}$$

and suggest for its solution some computational schemes.

We assume that the symbol $A(\xi)$ of the operator A satisfies the condition:

$$c_1(1 + |\xi|)^\alpha \leq |A(\xi)| \leq c_2(1 + |\xi|)^\alpha, \tag{6}$$

and it is well-known such a symbol admits factorization,

$$A(\xi) = A_+\left(\xi', \xi_m\right)A_-\left(\xi', \xi_m\right),$$

with respect to the last variable ξ_m with the index æ [1].

Since we know solvability conditions for pseudo-differential equations in $\mathbb{R}^m$ and $\mathbb{R}^m_+$ [1] we will select such discrete pseudo-differential operators which reserve all needed properties of their continuous analogues.

3.2.1 Equations in a whole space

Let P_h be a restriction operator on $h\mathbb{Z}^m$, i.e. for $u \in S(\mathbb{R}^m)$

$$(P_h u)(x) = \begin{cases} u(\tilde{x}), & x = \tilde{x} \in h\mathbb{Z}^m; \\ 0, & x \notin h\mathbb{Z}^m. \end{cases}$$

We tried this projector for simplest pseudo-differential operators, namely Calderon–Zygmund operators, these operators can be treated as pseudo-differential operators of order 0, and we obtained very acceptable results [3, 10–12]. But now we will use another restriction operator.

A construction for the restriction operator Q_h for functions $u \in S(\mathbb{R}^m)$ is the following. We take the Fourier transform $\tilde{u}(\xi)$, then its restriction on $\hbar\mathbb{T}^m$ and periodically continue it onto a whole $\mathbb{R}^m$. Further we apply the inverse discrete Fourier transform F^{-1}_d and obtain a discrete function which is denoted by $(Q_h u)(\tilde{x}), \quad \tilde{x} \in h\mathbb{Z}^m$. In our opinion the projector Q_h is more convenient than P_h although the projectors P_h and Q_h are almost the same according to the following result.

Lemma 1. *For* $u \in S(\mathbb{R}^m), \forall\beta > 0,$ *we have,*

$$|(P_h u)(\tilde{x}) - (Q_h u)(\tilde{x})| \leq Ch^\beta, \quad \forall\tilde{x} \in h\mathbb{Z}^m,$$

where the constant C depends on u only.

Further, the symbol $A_d(\xi)$ will be defined in the following way. We take a restriction of $A(\xi)$ on the cube $\hbar\mathbb{T}^m$ and periodically extend it onto a whole $\mathbb{R}^m$. We consider such h-operator as an approximate operator for A. So, to find a discrete solution for equation (3) for $D = \mathbb{R}^m$ we can use the following discrete equation:

$$A_d u_d = Q_h v. \tag{7}$$

Its solution is given by the formula,

$$u_d(\tilde{x}) = \frac{1}{(2\pi)^m} \int\limits_{\hbar\mathbb{T}^m} e^{i\tilde{x}\cdot\xi} A^{-1}(\xi)\tilde{v}(\xi)d\xi, \quad \tilde{x} \in h\mathbb{Z}^m,$$

so that we do not need to find an approximate solution for an infinite system of linear algebraic equations like [3, 10]. For our case we need to apply any kind of cubature formulas for calculating the latter integral and a cubature formula for calculating the Fourier transform $\tilde{v}(\xi)$.

According to Lemma 1 one can compare discrete and continuous solutions for enough smooth right-hand sides and symbols.

Theorem 1. *If the symbol $A(\xi)$ satisfies the condition and is infinitely differentiable on $\mathbb{R}^m$, u is a solution of the equation (4), u_d is a solution of the equation (5) then for $v \in S(\mathbb{R}^m)$ we have the following error estimate:*

$$|u(\tilde{x}) - u_d(\tilde{x})| \leq Ch^\beta, \quad \forall \tilde{x} \in h\mathbb{Z}^m,$$

for arbitrary $\beta > 0$.

Proofs for Lemma 1 and Theorem 1 are given in [17].

3.2.2 Equations in a half-space

If we put strong enough restrictions on a right-hand side and factorization elements then one can give a comparison between discrete and continuous solutions.

Lemma 2. *If $u \in S(\mathbb{R}^m)$ then the following estimate:*

$$\left| \left(F^{-1}P_{\xi'}\tilde{u}\right)(\tilde{x}) - \left(F_d^{-1}P_{\xi'}^{\text{per}}\widetilde{Q_h u})(\tilde{x})\right| \leq Ch^\beta, \quad \tilde{x} \in h\mathbb{Z}^m_+,$$

holds for $\forall\beta > 0$, and the constant C depends on u only.

Starting from Lemma 2 and the Theorem 1 we are able to compare discrete and continuous solutions in a half-space. Below we give this comparison under such conditions when a unique solution exists.

To formulate the following theorem we will describe how we need to choose a right-hand side for solving equation (3). We have the following solution of equation (5):

$$\tilde{u}(\xi) = A_+^{-1}(\xi)P_{\xi'}A_-^{-1}(\xi)\widetilde{\ell v}(\xi),$$

where $P_{\xi'} = \frac{1}{2}(I + H_{\xi'})$ is a projector defined by the classical Hilbert transform with respect to a variable ξ_m [1],

$$(H_{\xi'}\tilde{u})(\xi) = \frac{1}{\pi i} v.p. \int\limits_{-\infty}^{+\infty} \frac{\tilde{u}(\xi', \eta_m)d\eta_m}{\xi_m - \eta_m},$$

ℓv is a continuation of v from $\mathbb{R}^m_+$ into $\mathbb{R}^m$ in corresponding functional space. Since the right-hand side in equation (5) is defined in $h\mathbb{Z}^m_+$ then we choose $Q_h(\ell v)$ instead of ℓv_d to obtain the required estimate.

Theorem 2. *If the symbol $A(\xi)$ satisfies the condition (6) and is infinitely differentiable in $\mathbb{R}^m$ with the factors $A_\pm(\xi)$, u is a solution of the equation (5), u_d is a solution of the equation (3) then for $v \in S(\mathbb{R}^m)$ we have the following error estimate:*

$$|u(\tilde{x}) - u_d(\tilde{x})| \leq Ch^\beta, \quad \forall \tilde{x} \in h\mathbb{Z}^m_+,$$

for arbitrary $\beta > 0$.

One can find proofs for Lemma 2 and Theorem 2 in [17].

3.3 Non-trivial case

We have non-uniqueness of a solution for equation (3) for the case $æ - s = n + \delta, n \in \mathbb{N}, |\delta| < 1/2$. We consider here the case $n = 1$.

To obtain the unique solution one needs some additional conditions. Discrete analogues of Dirichlet or Neumann conditions give a very simple case. We will consider here the discrete Dirichlet condition:

$$u_d|_{\tilde{x}_m=0} = g_d(\tilde{x'}), \tag{8}$$

where g_d is a given function of a discrete variable in the discrete hyper-plane hZ^{m-1}.

The condition (8) in Fourier images takes the form:

$$\int_{-\hbar\pi}^{\hbar\pi} \tilde{u}_d(\xi', \xi_m)\mathrm{d}\xi_m = \tilde{g}_d(\xi'),$$

and according to the previous theorem we obtain the following integral equation with respect to the unknown $\tilde{c}_0(\xi')$,

$$\int_{-\hbar\pi}^{\hbar\pi} \tilde{c}_0\left(\xi'\right)A^{-1}_{d,+}\left(\xi', \xi_m\right)\mathrm{d}\xi_m = \tilde{f}_d\left(\xi'\right),$$

where we have used the following notation,

$$\tilde{f}_d(\xi') = \tilde{g}_d(\xi') - \int_{-\hbar\pi}^{\hbar\pi} A^{-1}_{d,+}(\xi', \xi_m)X_1(\xi', \xi_m)P^{\mathrm{per}}_{\xi'}(X_1^{-1}A^{-1}_{d,-}\widetilde{\ell v_d})(\xi)\mathrm{d}\xi_m,$$

where $X_1(\xi)$ is a polynomial of order 1 of variables $\hat{\zeta}_k = \hbar(e^{-ih\xi_k} - 1)$, $k = 1,\ldots, m$ from the class E_1.

Let us denote,

$$\int_{-\hbar\pi}^{\hbar\pi} A^{-1}_{d,+}(\xi', \xi_m)\mathrm{d}\xi_m \equiv b_d(\xi'),$$

and assuming that $b_d(\xi') \neq 0$ we will find,

$$\tilde{c}_0(\xi') = b_d^{-1}(\xi')\tilde{f}_d(\xi').$$

Then the solution of the problem (3), (8) is the following,

$$\tilde{u}_d(\xi) = \tilde{u}_d(\xi) = \tilde{A}^{-1}_{d,+}(\xi)X_1(\xi)P^{\mathrm{per}}_{\xi'}(X_1^{-1}(\xi)\tilde{A}^{-1}_{d,-}(\xi)\widetilde{\ell v}_d(\xi)) + b_d^{-1}(\xi')\tilde{f}_d(\xi')A^{-1}_{d,+}(\xi', \xi_m).$$

Thus, we obtain the following result.

Theorem 3. *Discrete boundary value problem (3), (8) is uniquely solvable in the space $H^s(h\mathbb{Z}^m_+)$ for arbitrary right-hand side $v_d \in H_0^{s-\alpha}(h\mathbb{Z}^m_+)$ and arbitrary boundary function $g_d \in H^{s-1/2}(h\mathbb{Z}^{m-1})$.*

If the right-hand side is zero, i.e. $v_d \equiv 0$, then the formula for the solution is very simplified and looks as follows:

$$\tilde{u}_d(\xi) = b_d^{-1}(\xi')\tilde{g}_d(\xi')A^{-1}_{d,+}(\xi', \xi_m),$$

and after inverse discrete Fourier transform it will be the following,

$$u_d(\tilde{x'}, \tilde{x}_m) = \sum_{\tilde{y}' \in h\mathbb{Z}^{m-1}} G_d(\tilde{x}' - \tilde{y}', \tilde{x}_m)g_d(\tilde{y}')h^{m-1}, \tag{9}$$

where the function $G_d(\tilde{x})$ of a discrete variable is defined as inverse discrete Fourier transform of the function,

$$b_d^{-1}(\xi')A^{-1}_{d,+}(\xi', \xi_m).$$

The formula (9) is a discrete analogue of Poisson formula for the Dirichlet problem in a half-space.

4 Comparison

To obtain some comparison between discrete and continuous solutions we will remind how the continuous solution looks. The continuous analogue of the discrete boundary value problem is the following:

$$(Au)(x) = 0, \quad x \in \mathbb{R}^m_+, \tag{10}$$

$$u(x', 0) = g(x'), \quad x' \in \mathbb{R}^{m-1}. \tag{11}$$

If the index of factorization equals to æ and æ$-s = 1 + \delta$, $|\delta| < 1/2$ then the unique solution for the problem (10), (11) is constructed by the similar formula:

$$\tilde{u}(\xi) = b^{-1}(\xi')\tilde{g}(\xi')A_+^{-1}(\xi', \xi_m),$$

where,

$$b(\xi') = \int_{-\infty}^{+\infty} A_+^{-1}(\xi', \xi_m)d\xi_m,$$

assuming that $b(\xi') \neq 0, \forall \xi' \in \mathbb{R}^{m-1}$. Let us note that this is simplest variant of Shapiro–Lopatinskii condition [1].

We have the following discrete solution:

$$\tilde{u}_d(\xi) = b_d^{-1}(\xi')\tilde{g}_d(\xi')A_{d,+}^{-1}(\xi', \xi_m),$$

in which we choose special approximations. We take $g_d = Q_h g$ and $A_{d,\pm}(\xi', \xi_m)$ we take as restrictions of $A_\pm(\xi', \xi_m)$ on $\hbar\mathbb{T}^m$. Then the periodic symbol,

$$A_d(\xi) = A_{d,+}\left(\xi', \xi_m\right)A_{d,-}\left(\xi', \xi_m\right),$$

satisfies all conditions of periodic factorization with the same index æ. Moreover, $\tilde{g}_d(\xi')$ and $A_{d,+}(\xi', \xi_m)$ coincide with $\tilde{g}(\xi')$ and $A_+(\xi', \xi_m)$ respectively on $\hbar\mathbb{T}^m$.

Theorem 4. *Let æ > 1. If $\tilde{g}(\xi)$ is a bounded function then a comparison between solutions of problems (3), (8) and (10), (11) is given in the following way,*

$$|\tilde{u}(\xi) - \tilde{u}_d(\xi)| \leq Ch^{-1}, \xi \in \hbar\mathbb{T}^m.$$

Proof. For this case we have exact formulas for both continuous and discrete solutions. We will estimate the difference,

$$\tilde{u}(\xi) - \tilde{u}_d(\xi), \xi \in \hbar\mathbb{T}^m.$$

we have,

$$\tilde{u}(\xi) - \tilde{u}_d(\xi) = b^{-1}\left(\xi'\right)\tilde{g}\left(\xi'\right)A_+^{-1}\left(\xi', \xi_m\right) - b_d^{-1}\left(\xi'\right)\tilde{g}_d\left(\xi'\right)A_{d,+}^{-1}\left(\xi', \xi_m\right) = (b^{-1}(\xi') - b_d^{-1}(\xi'))\tilde{g}_d(\xi')A_{d,+}^{-1}(\xi', \xi_m), \quad \xi \in \hbar\mathbb{T}^m,$$

so that we need to estimate $b^{-1}(\xi') - b_d^{-1}(\xi')$. Since,

$$\left|b^{-1}\left(\xi'\right) - b_d^{-1}\left(\xi'\right)\right| = \frac{\left|b\left(\xi'\right) - b_d\left(\xi'\right)\right|}{\left|b\left(\xi'\right)\right|\left|b_d\left(\xi'\right)\right|} \leq C\left|b\left(\xi'\right) - b_d\left(\xi'\right)\right|,$$

because $\inf|b(\xi')| \geq c_3, \inf|b_d(\xi')| \geq c_4$ then we will estimate the latter difference. Simplest considerations lead to the following estimate,

$$|b(\xi') - b_d(\xi')| = \left|\int_{-\infty}^{+\infty} A_+^{-1}(\xi', \xi_m)\mathrm{d}\xi_m - \int_{-\hbar\pi}^{\hbar\pi} A_{d,+}^{-1}(\xi', \xi_m)\mathrm{d}\xi_m\right| = \left|\left(\int_{-\infty}^{-\hbar\pi} + \int_{\hbar\pi}^{+\infty}\right)A_+^{-1}(\xi', \xi_m)\mathrm{d}\xi_m\right|.$$

We will estimate one integral only, the second one is almost the same,

$$\int_{\hbar\pi}^{+\infty} |A_+^{-1}(\xi', \xi_m)| \mathrm{d}\xi_m \leq c_5 \int_{\hbar\pi}^{+\infty} (1 + |\xi'| + |\xi_m|)^{-æ} \mathrm{d}\xi_m = \frac{c_5}{-1} (1 + |\xi'| + \hbar\pi)^{1-æ} \leq c_6 h^{-1}.$$

Therefore if $\tilde{g}(\xi')$ is a bounded function we have the required estimate.

Conclusion

This paper is one of first steps for studying discrete boundary value problems and their connections with classical theory of boundary value problems for elliptic pseudo-differential equations. We intend to study more general situations in forthcoming papers and to obtain approximation estimates for comparison of discrete and continuous solutions.

Acknowledgements

Research supported by the State contract of the Russian Ministry of Education and Science (contract No 1.7311.2017/8.9).

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Cite this article as: Tarasova O.A & Vasilyev V.B 2019. To the theory of discrete boundary value problems. 4open, **2**, 17.