4open 2019, **2**, 24
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https://doi.org/10.1051/fopen/2019021

Special Issue: Advances in Researches of Quaternion Algebras
Guest Editor: Ivan Kyrchei

Available online at:
www.4open-sciences.org

RESEARCH ARTICLE OPEN ACCESS

Cramer's rules for the system of quaternion matrix equations with η-Hermicity

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Received 10 January 2019, Accepted 6 June 2019

Abstract – The system of two-sided quaternion matrix equations with η-Hermicity, $\mathbf{A}_1\mathbf{X}\mathbf{A}_1^{\eta*} = \mathbf{C}_1$, $\mathbf{A}_2\mathbf{X}\mathbf{A}_2^{\eta*} = \mathbf{C}_2$ is considered in the paper. Using noncommutative row-column determinants previously introduced by the author, determinantal representations (analogs of Cramer's rule) of a general solution to the system are obtained. As special cases, Cramer's rules for an η-Hermitian solution when $\mathbf{C}_1 = \mathbf{C}_1^{\eta*}$ and $\mathbf{C}_2 = \mathbf{C}_2^{\eta*}$ and for an η-skew-Hermitian solution when $\mathbf{C}_1 = -\mathbf{C}_1^{\eta*}$ and $\mathbf{C}_2 = -\mathbf{C}_2^{\eta*}$ are also explored.

Keywords: Generalized inverse, Noncommutative determinant, Quaternion matrix, System of matrix equations, Cramer rule, η-Hermicity

2000 AMS subject classifications: 15A24, 15A15, 15A09, 15B33

Introduction

In the whole article, the notation $\mathbb{R}$ is reserved for the real number field and $\mathbb{H}^{m\times n}$ stands for the set of all $m \times n$ matrices over the quaternion skew field

$$\mathbb{H} = \left\{h_0 + h_1\mathbf{i} + h_2\mathbf{j} + h_3\mathbf{k} | \mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1, h_0, h_1, h_2, h_3 \in \mathbb{R}\right\}.$$

$\mathbb{H}_r^{m\times n}$ specifies its subset of matrices with a rank r. For given $h = h_0 + h_1\mathbf{i} + h_2\mathbf{j} + h_3\mathbf{k} \in \mathbb{H}$, the conjugate of h is $\overline{h} = a_0 - h_1\mathbf{i} - h_2\mathbf{j} - h_3\mathbf{k}$. For given $\mathbf{A} \in \mathbb{H}^{n\times m}$, $\mathbf{A}^*$ represents the conjugate transpose (Hermitian adjoint) matrix of $\mathbf{A}$. The matrix $\mathbf{A} \in \mathbb{H}^{n\times n}$ is Hermitian if $\mathbf{A}^* = \mathbf{A}$. $\mathbf{A}^\dagger$ means the Moore–Penrose inverse of $\mathbf{A} \in \mathbb{H}^{n\times m}$, i.e. the exclusive matrix $\mathbf{X}$ satisfying the following four equations

$$(1)\,\mathbf{AXA} = \mathbf{A}, \quad (2)\,\mathbf{XAX} = \mathrm{X}, \quad (3)\,(\mathbf{AX})^* = \mathbf{AX}, \quad (4)\,(\mathbf{XA})^* = \mathbf{XA}.$$

Quaternions have ample use in diverse areas such, such as color imaging and computer science [1–5], fluid mechanics [6, 7], quantum mechanics [8, 9], the attitude orientation and spatial rigid body dynamics [10–12], signal processing [13–15], etc.

The research of matrix equations have both applied and theoretical importance. Many authors explored the system of two-sided matrix equations

$$\begin{cases} \mathbf{A}_1\mathbf{X}\mathbf{B}_1 = \mathbf{C}_1, \\ \mathbf{A}_2\mathbf{X}\mathbf{B}_2 = \mathbf{C}_2. \end{cases} \tag{1}$$

over the field of complex numbers, the quaternion skew field, etc. (see, e.g. [16–21]). In this paper, the following system of quaternion matrix equations with η-Hermicity are considered,

$$\begin{cases} \mathbf{A}_1\mathbf{X}\mathbf{A}_1^{\eta*} = \mathbf{C}_1, \\ \mathbf{A}_2\mathbf{X}\mathbf{A}_2^{\eta*} = \mathbf{C}_2. \end{cases} \tag{2}$$

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Definition 1.1. [22–24] A matrix $\mathbf{A} \in \mathbb{H}^{n\times n}$ is known to be *η-Hermitian* and *η-skew-Hermitian* if $\mathbf{A} = \mathbf{A}^{\eta*} = -\eta\mathbf{A}^*\eta$ and $\mathbf{A} = -\mathbf{A}^{\eta*} = \eta\mathbf{A}^*\eta$, respectively, where $\eta \in \{\mathbf{i},\mathbf{j},\mathbf{k}\}$.

Convergence analysis in statistical signal processing and linear modeling [14, 15, 23] are some fields in which the applications of η-Hermitian matrices matrices can be viewed. The singular value decomposition of the η-Hermitian matrix was examined in [22]. Very recently, Liu [25] determined η-skew-Hermitian solutions to some classical matrix equations and, among them, the generalized Sylvester-type matrix equation:

$$\mathbf{AXA}^{\eta*} + \mathbf{BYB}^{\eta*} = \mathbf{C}. \tag{3}$$

Note that in [25], the term "η-anti-Hermitian" has been used instead "η-skew-Hermitian". He and Wang [26] gave the general solution of

$$\mathbf{AX} + (\mathbf{AX})^{\eta*} + \mathbf{BYB}^{\eta*} + \mathbf{CZC}^{\eta*} = \mathbf{D},$$

bearing η-Hermicity over $\mathbb{H}$ by expressing it's general η-Hermitian solution in terms of the Moore–Penrose inverses. An iterative algorithm for determining η (-skew)-Hermitian least-squares solutions to the quaternion matrix equation (3) was established in [27]. For more related papers on η-Hermicity and its generalization, ϕ-Hermicity, one may refer to [28–38].

In this paper, we construct novel explicit determinantal representation formulas (an analog of Cramer's rule) of the general and η-(skew-)Hermitian solutions to the system (2), by using determinantal representations of the Moore–Penrose matrix that was obtained within in the framework of the theory of row-column noncommutative determinants. According to our best of knowledge, our Cramer's rule proposed is a unique direct method to compute the η-(skew-)Hermitian solutions to quaternion matrix equations unlike other similar works (see, e.g. [24–26, 29, 32]), where obtained explicit forms of solutions have mostly only theoretical significance.

In contrast to the inverse matrix that has a definitely determinantal representation in terms of cofactors, for generalized inverse matrices, in particular, Moore–Penrose matrices, there exist different determinantal representations even for matrices with real or complex entries as a result of the search of their more applicable explicit expressions (for the Moore–Penrose matrix, see, e.g., [39–41]). For quaternion matrices, in view of the noncommutativity of quaternions, the problem of the determinantal representation of generalized inverse matrices remained open for a long time and only now can be solved due to the theory of row-column determinants which were introduced in [42, 43].

Currently, applying of row-column determinants to determinantal representations of various generalized inverses have been derived by the author (see, e.g. [44–57]) and other researchers (see, e.g. [58–61]). In particular, determinantal representations of systems like to (1) have been recently explored in [53, 55, 56, 61].

The remainder of the paper is directed as follows. In Section 2, we start with preliminaries in general properties generalized inverses, projectors, and η-matrices in Section 2.1, and in the theory of row-column determinants and determinantal representations of the Moore–Penrose inverses of a quaternion matrix, its Hermitian adjoint and η-Hermitian adjoint matrices in Section 2.2. Determinantal representations of a general, η-Hermitian and η-skew-Hermitian solutions to the system (2) are derived in Section 3. Finally, the conclusion is drawn in Section 4.

Preliminaries: Determinantal representations of solutions to quaternion matrix equations

General properties generalized inverses, projectors, and η-matrices

We begin with some famous results on generalized inverses and projectors inducted by them which will be used in the remaining part of this paper.

Lemma 2.1. [26] *Let* $\mathbf{A} \in \mathbb{H}^{m\times n}$. *Then*

(1) $(\mathbf{A}^{\eta})^{\dagger} = (\mathbf{A}^{\dagger})^{\eta}, (\mathbf{A}^{\eta*})^{\dagger} = (\mathbf{A}^{\dagger})^{\eta*}$.

(2) $\operatorname{rank}\mathbf{A} = \operatorname{rank}\mathbf{A}^{\eta*} = \operatorname{rank}\mathbf{A}^{\eta} = \operatorname{rank}\mathbf{A}^{\eta}\mathbf{A}^{\eta*} = \operatorname{rank}(\mathbf{A}^{\eta*}\mathbf{A}^{\eta})$.

(3) $(\mathbf{A}^{\dagger}\mathbf{A})^{\eta*} = \mathbf{A}^{\eta*}(\mathbf{A}^{\dagger})^{\eta*} = (\mathbf{A}^{\dagger}\mathbf{A})^{\eta} = (\mathbf{A}^{\dagger})^{\eta}\mathbf{A}^{\eta}$.

(4) $(\mathbf{A}\mathbf{A}^{\dagger})^{\eta*} = (\mathbf{A}^{\dagger})^{\eta*}\mathbf{A}^{\eta*} = (\mathbf{A}\mathbf{A}^{\dagger})^{\eta} = \mathbf{A}^{\eta}(\mathbf{A}^{\dagger})^{\eta}$.

(5) $\mathbf{L}_A^{\eta*} = -\eta(\mathbf{L}_A)\eta = \mathbf{L}_A^{\eta} = \mathbf{L}_{A^{\eta}} = \mathbf{R}_{A^{\eta*}}$.

(6) $\mathbf{R}_A^{\eta*} = -\eta(\mathbf{R}_A)\eta = \mathbf{R}_A^{\eta} = \mathbf{L}_{A^{\eta*}} = \mathbf{R}_{A^{\eta}}$.

Lemma 2.2. [72] *Let* $\mathbf{A}$, $\mathbf{B}$ *and* $\mathbf{C}$ *be given matrices with right sizes over* $\mathbb{H}$. *Then*

(1) $\mathbf{A}^\dagger = (\mathbf{A}^*\mathbf{A})^\dagger\mathbf{A}^* = \mathbf{A}^*(\mathbf{A}\mathbf{A}^*)^\dagger$.
(2) $\mathbf{L}_A = \mathbf{L}_A^2 = \mathbf{L}_A^*, \mathbf{R}_A = \mathbf{R}_A^2 = \mathbf{R}_A^*$.
(3) $\mathbf{L}_A(\mathbf{B}\mathbf{L}_A)^\dagger = (\mathbf{B}\mathbf{L}_A)^\dagger, (\mathbf{R}_A\mathbf{C})^\dagger\mathbf{R}_A = (\mathbf{R}_A\mathbf{C})^\dagger$.

Remark 2.1. For any $\eta_l \in \{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ for all $l = 1, 2, 3$, and $q = q_0 + q_1\eta_1 + q_2\eta_2 + q_3\eta_3$, we denote

$$q^{\eta_1} := -\eta_1 q\eta_1 = q_0 + q_1\eta_1 - q_2\eta_2 - q_3\eta_3,$$
$$q^{-\eta_1} := \eta_1 q\eta_1 = -q_0 - q_1\eta_1 + q_2\eta_2 + q_3\eta_3.$$

So, elements of the main diagonal of an η_1-Hermitian matrix $\mathbf{A} = \mathbf{A}^{\eta_1 *} = \left(a_{ij}^{\eta_1 *}\right)$ should be as follows

$$a_{ii}^{\eta_1 *} = a_0 + a_2\eta_2 + a_3\eta_3,$$

and a pair of elements which are symmetric with respect to the main diagonal can be represented as

$$a_{ij}^{\eta_1 *} = a_0 + a_1\eta_1 + a_2\eta_2 + a_3\eta_3,$$
$$a_{ji}^{\eta_1 *} = a_0 - a_1\eta_1 + a_2\eta_2 + a_3\eta_3.$$

Similarly, elements of the main diagonal of an η_1-skew-Hermitian matrix $\mathbf{A} = -\mathbf{A}^{\eta_1 *} = \left(a_{ij}^{-\eta_1 *}\right)$ should be as follows

$$a_{ii}^{-\eta_1 *} = a_1\eta_1,$$

and a pair of elements which are symmetric with respect to the main diagonal can be represented as

$$a_{ij}^{-\eta_1 *} = a_0 + a_1\eta_1 + a_2\eta_2 + a_3\eta_3,$$
$$a_{ji}^{-\eta_1 *} = -a_0 + a_1\eta_1 - a_2\eta_2 - a_3\eta_3.$$

where $a_l \in \mathbb{R}$ for all $l = 0,\ldots, 3$.

Determinantal representations of generalized inverses and of solutions to some quaternion matrix equations

Through the non-commutativity of the quaternion skew field, determining of the determinant with noncommutative entries (it is also called a noncommutative determinant) is not so trivial (see, e.g. [62, 63]). There are several versions of the definition of noncommutative determinants (see, e.g., [64–69]). But, it is proved in [70], if all functional properties of determinant over a ring are satisfied, then it takes on a value in its commutative subset only. In particular, it means that such determinant can not be expanded by cofactors along an arbitrary row or column. To avoid these difficulties, for $\mathbf{A} \in \mathbb{H}^{n\times n}$, we define n row determinants and n column determinants which are not owning of all functional properties that could be inherent to the usual determinant.

Suppose S_n is the symmetric group on the set $I_n = \{1, \ldots, n\}$.

Definition 2.2. [42] The ith row determinant *of* $\mathbf{A} = (a_{ij}) \in \mathbb{H}^{n\times n}$ is called by setting for all $i = 1, \ldots, n$,

$$\operatorname{rdet}_i \mathbf{A} = \sum_{\sigma\in S_n} (-1)^{n-r} (a_{i i_{k_1}} a_{i_{k_1} i_{k_1+1}} \ldots a_{i_{k_1+l_1} i}) \ldots (a_{i_{k_r} i_{k_r+1}} \ldots a_{i_{k_r+l_r} i_{k_r}}),$$
$$\sigma = (i i_{k_1} i_{k_1+1} \ldots i_{k_1+l_1})(i_{k_2} i_{k_2+1} \ldots i_{k_2+l_2}) \ldots (i_{k_r} i_{k_r+1} \ldots i_{k_r+l_r}),$$

where σ is the left-ordered permutation. It means that its first cycle from the left starts with i, other cycles start from the left with the minimal of all the integers which are contained in it,

$$i_{k_t} < i_{k_t+s} \quad \text{for all} \quad t = 2, \ldots, r, \quad s = 1, \ldots, l_t,$$

and the order of disjoint cycles (except for the first one) is strictly conditioned by increase from left to right of their first elements, $i_{k_2} < i_{k_3} < \cdots < i_{k_r}$.

Definition 2.3. [42] The jth column determinant of $\mathbf{A} = (a_{ij}) \in \mathbb{H}^{n\times n}$ is called by setting for all $j = 1, \ldots, n$,

$$\operatorname{cdet}_j \mathbf{A} = \sum_{\tau\in S_n} (-1)^{n-r} (a_{j_{k_r} j_{k_r+l_r}} \ldots a_{j_{k_r+1} j_{k_r}}) \ldots (a_{j j_{k_1+l_1}} \ldots a_{j_{k_1+1} j_{k_1}} a_{j_{k_1} j}),$$
$$\tau = \left(j_{k_r+l_r} \ldots j_{k_r+1} j_{k_r}\right) \ldots \left(j_{k_2+l_2} \ldots j_{k_2+1} j_{k_2}\right)\left(j_{k_1+l_1} \ldots j_{k_1+1} j_{k_1} j\right),$$

where τ is the right-ordered permutation. It means that its first cycle from the right starts with j, other cycles start from the right with the minimal of all the integers which are contained in it,

$$j_{k_t} < j_{k_t+s} \quad \text{for all} \quad t = 2, \ldots, r, \quad s = 1, \ldots, l_t,$$

and the order of disjoint cycles (except for the first one) is strictly conditioned by increase from right to left of their first elements, $j_{k_2} < j_{k_3} < \cdots < j_{k_r}$.

Remark 2.4. So, for a 2×2-matrix with quaternion settings $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, we have the four (row-column) determinants

$$\begin{aligned} \mathrm{rdet}_1\mathbf{A} &= a_{11}a_{22} - a_{12}a_{21}, \quad \mathrm{rdet}_2\mathbf{A} = a_{22}a_{11} - a_{21}a_{12}, \\ \mathrm{cdet}_1\mathbf{A} &= a_{22}a_{11} - a_{12}a_{21}, \quad \mathrm{cdet}_2\mathbf{A} = a_{11}a_{22} - a_{21}a_{12}. \end{aligned}$$

Since $a_{ij} \in \mathbb{H}$ for all $i, j = 1, 2$, they are not equal to each others, in general.

We state some properties of row-column determinants needed below.

Lemma 2.3. [42] *If the ith row of* $\mathbf{A} \in \mathbb{H}^{n\times n}$ *is a left linear combination of other row vectors, i.e.* $a_{i.} = \alpha_1\mathbf{b}_1 + \cdots + \alpha_k\mathbf{b}_k$, *where* $\alpha_l \in \mathbb{H}$ *and* $\mathbf{b}_l \in \mathbb{H}^{1\times n}$ *for all* $l = 1, \ldots, k$ *and* $i = 1, \ldots, n$, *then*

$$\mathrm{rdet}_i\, \mathbf{A}_{i.}(\alpha_1\mathbf{b}_1 + \cdots + \alpha_k\mathbf{b}_k) = \sum_l \alpha_l \mathrm{rdet}_i\, \mathbf{A}_{i.}(\mathbf{b}_l).$$

Lemma 2.4. [42] *If the jth column of* $\mathbf{A} \in \mathbb{H}^{m\times n}$ *is a right linear combination of other column vectors, i.e.* $a_{.j} = \mathbf{b}_1\alpha_1 + \cdots + \mathbf{b}_k\alpha_k$, *where* $\alpha_l \in \mathbb{H}$ *and* $\mathbf{b}_l \in \mathbb{H}^{n\times 1}$ *for all* $l = 1, \ldots, k$ *and* $j = 1, \ldots, n$, *then*

$$\mathrm{cdet}_j\, \mathbf{A}_{.j}(\mathbf{b}_1\alpha_1 + \cdots + \mathbf{b}_k\alpha_k) = \sum_l \mathrm{cdet}_j\, \mathbf{A}_{.j}(\mathbf{b}_l)\alpha_l.$$

Lemma 2.5. [43] *Let* $\mathbf{A} \in \mathbb{H}^{n\times n}$. *Then* $cdet_i\, \mathbf{A}^* = \overline{rdet_i\, \mathbf{A}}$, $rdet_i\, \mathbf{A}^* = \overline{cdet_i\, \mathbf{A}}$ *for all* $i = 1, \ldots, n$.

Since by Definitions 2.2 and 2.3 for $\mathbf{A} \in \mathbb{H}^{n\times n}$

$$\begin{aligned} \mathrm{rdet}_i\, \mathbf{A}^\eta &= \mathrm{rdet}_i\,(-\eta\mathbf{A}\eta) = -\eta(\mathrm{rdet}_i\, \mathbf{A})\eta, \\ \mathrm{cdet}_i\, \mathbf{A}^\eta &= \mathrm{cdet}_i\,(-\eta\mathbf{A}\eta) = -\eta(\mathrm{cdet}_i\, \mathbf{A})\eta, \\ \mathrm{rdet}_i\,(-\mathbf{A}^\eta) &= \mathrm{rdet}_i\,(\eta\mathbf{A}\eta) = (-1)^{n-1}\eta(\mathrm{rdet}_i\, \mathbf{A})\eta, \\ \mathrm{cdet}_i\,(-\mathbf{A}^\eta) &= \mathrm{cdet}_i\,(\eta\mathbf{A}\eta) = (-1)^{n-1}\eta(\mathrm{cdet}_i\, \mathbf{A})\eta, \end{aligned}$$

for all $i = 1, \ldots, n$, then, due to Lemma 2.5, the next lemma follows immediately.

Lemma 2.6. *Let* $\mathbf{A} \in \mathbb{H}^{n\times n}$. *Then*

$$\begin{aligned} \mathrm{rdet}_i\, \mathbf{A}^{\eta *} &= -\eta(\overline{\mathrm{cdet}_i\, \mathbf{A}})\eta, \quad \mathrm{cdet}_i\, \mathbf{A}^{\eta *} = -\eta(\overline{\mathrm{rdet}_i\, \mathbf{A}})\eta, \\ \mathrm{rdet}_i\,(-\mathbf{A}^{\eta *}) &= (-1)^{n-1}\eta(\overline{\mathrm{cdet}_i\, \mathbf{A}})\eta, \quad \mathrm{cdet}_i\,(-\mathbf{A}^{\eta *}) = (-1)^{n-1}\eta(\overline{\mathrm{rdet}_i\, \mathbf{A}})\eta, \end{aligned}$$

for all $i = 1, \ldots, n$.

Remark 2.5. Since [42] for Hermitian $\mathbf{A}$ *we have*

$$\mathrm{rdet}_1\mathbf{A} = \cdots = \mathrm{rdet}_n\mathbf{A} = \mathrm{cdet}_1\mathbf{A} = \cdots = \mathrm{cdet}_n\mathbf{A} \in \mathbb{R},$$

the determinant of a Hermitian matrix is called by setting $\det \mathbf{A} := \mathrm{rdet}_i\, \mathbf{A} = \mathrm{cdet}_i\, \mathbf{A}$ for any $i = 1, \ldots, n$.

Its properties have been completely studied in [43]. In particular, from them it follows the definition of the *determinantal rank* of a quaternion matrix $\mathbf{A}$ as the largest possible size of nonzero principal minors of its corresponding Hermitian matrices, i.e. $\mathrm{rank}\mathbf{A} = \mathrm{rank}(\mathbf{A}^*\mathbf{A}) = \mathrm{rank}(\mathbf{A}\mathbf{A}^*)$.

For determinantal representations of the Moore–Penrose inverse, we use the following notations. Let $\alpha := \{\alpha_1, \ldots, \alpha_k\} \subseteq \{1, \ldots, m\}$ and $\beta := \{\beta_1, \ldots, \beta_k\} \subseteq \{1, \ldots, n\}$ be subsets with $1 \le k \le \min\{m, n\}$. By $\mathbf{A}^\alpha_\beta$ denote a submatrix of $\mathbf{A} \in \mathbb{H}^{m\times n}$ with rows and columns indexed by α and β, respectively. Then, $\mathbf{A}^\alpha_\alpha$ is a principal submatrix of $\mathbf{A}$ with rows and columns indexed by α. Moreover, for Hermitian $\mathbf{A}$, $|\mathbf{A}|^\alpha_\alpha$ is the principal minor of $\det \mathbf{A}$. Suppose that,

$$L_{k,n} := \{\, \alpha : \alpha = (\alpha_1, \ldots, \alpha_k), \quad 1 \le \alpha_1 < \cdots < \alpha_k \le n \,\},$$

stands for the collection of strictly increasing sequences of $1 \leq k \leq n$ integers chosen from $\{1, \dots, n\}$. For fixed $i \in \alpha$ and $j \in \beta$, put $I_{r,m}\{i\} := \{\alpha : \alpha \in L_{r,m}, i \in \alpha\}$, $J_{r,n}\{j\} := \{\beta : \beta \in L_{r,n}, j \in \beta\}$.

By $\mathbf{a}_{.j}$ and $\mathbf{a}^*_{.j}$, $\mathbf{a}_{i.}$ and $\mathbf{a}^*_{i.}$ denote the jth columns and the ith rows of $\mathbf{A}$ and $\mathbf{A}^*$, respectively. Suppose $\mathbf{A}_{i.}(\mathbf{b})$ and $\mathbf{A}_{.j}(\mathbf{c})$ stand for the matrices obtained from $\mathbf{A}$ by replacing its ith row with the row $\mathbf{b}$ and its jth column with the column $\mathbf{c}$, respectively.

Theorem 2.6. [44] *If $\mathbf{A} \in \mathbb{H}^{m\times n}_r$, then its Moore–Penrose inverse $\mathbf{A}^\dagger = \left(a^\dagger_{ij}\right) \in \mathbb{H}^{n\times m}$ is determined as follows*

$$a^\dagger_{ij} = \tag{4}$$
$$= \frac{\sum\limits_{\beta\in J_{r,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.i}\left(\mathbf{a}^*_{.j}\right)\right)^\beta_\beta}{\sum\limits_{\beta\in J_{r,n}} \left|\mathbf{A}^*\mathbf{A}\right|^\beta_\beta}$$

$$= \frac{\sum\limits_{\alpha\in I_{r,m}\{j\}} \operatorname{rdet}_j\left(\left(\mathbf{A}\mathbf{A}^*\right)_{j.}\left(\mathbf{a}^*_{i.}\right)\right)^\alpha_\alpha}{\sum\limits_{\alpha\in I_{r,m}} \left|\mathbf{A}\mathbf{A}^*\right|^\alpha_\alpha}. \tag{5}$$

Remark 2.7. For an arbitrary full-rank matrix $\mathbf{A} \in \mathbb{H}^{m\times n}_r$, a row-vector $\mathbf{b} \in \mathbb{H}^{1\times m}$, and a column-vector $\mathbf{c} \in \mathbb{H}^{n\times 1}$, we assume that for all $i = 1, \dots, m$, $j = 1, \dots, n$,

- if rank $\mathbf{A} = n$, then in (4)

$$\operatorname{cdet}_j\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.j}(\mathbf{c})\right) = \sum_{\beta\in J_{n,n}\{j\}} \operatorname{cdet}_j\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.j}(\mathbf{c})\right)^\beta_\beta,$$
$$\det\left(\mathbf{A}^*\mathbf{A}\right) = \sum_{\beta\in J_{n,n}} \left|\mathbf{A}^*\mathbf{A}\right|^\beta_\beta;$$

- if rank $\mathbf{A} = m$, then in (5)

$$\operatorname{rdet}_i\left(\left(\mathbf{A}\mathbf{A}^*\right)_{i.}(\mathbf{b})\right) = \sum_{\alpha\in I_{m,m}\{i\}} \operatorname{rdet}_i\left(\left(\mathbf{A}\mathbf{A}^*\right)_{i.}(\mathbf{b})\right)^\alpha_\alpha,$$
$$\det\left(\mathbf{A}\mathbf{A}^*\right) = \sum_{\alpha\in I_{m,m}} \left|\mathbf{A}\mathbf{A}^*\right|^\alpha_\alpha.$$

Corollary 2.1. *If $\mathbf{A} \in \mathbb{H}^{m\times n}_r$, then the Moore–Penrose inverse $\left(\mathbf{A}^\eta\right)^\dagger = \left(a^{\eta\dagger}_{ij}\right) \in \mathbb{H}^{n\times m}$ have the following determinantal representations:*

$$a^{\eta\dagger}_{ij} = -\eta\frac{\sum\limits_{\beta\in J_{r,n}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}^*\mathbf{A}\right)_{.i}\left(\mathbf{a}^*_{.j}\right)\right)^\beta_\beta}{\sum\limits_{\beta\in J_{r,n}} \left|\mathbf{A}^*\mathbf{A}\right|^\beta_\beta}\eta = -\eta\frac{\sum\limits_{\alpha\in I_{r,m}\{j\}} \operatorname{rdet}_j\left(\left(\mathbf{A}\mathbf{A}^*\right)_{j.}\left(\mathbf{a}^*_{i.}\right)\right)^\alpha_\alpha}{\sum\limits_{\alpha\in I_{r,m}} \left|\mathbf{A}\mathbf{A}^*\right|^\alpha_\alpha}\eta.$$

Remark 2.8. Since $\left(\mathbf{A}^*\right)^\dagger = \left(\mathbf{A}^\dagger\right)^*$, then we can use the denotation $\mathbf{A}^{\dagger,*} := \left(\mathbf{A}^*\right)^\dagger$. By Lemma 2.5, for the Hermitian adjoint matrix $\mathbf{A}^* \in \mathbb{H}^{n\times m}_r$, its Moore–Penrose inverse $\left(\mathbf{A}^*\right)^\dagger = \left(\left(a^*_{ij}\right)^\dagger\right) \in \mathbb{H}^{m\times n}$ can be expressed as

$$\left(a^*_{ij}\right)^\dagger = \overline{\left(a_{ji}\right)}^\dagger = \frac{\sum\limits_{\alpha\in I_{r,n}\{j\}} \operatorname{rdet}_j\left(\left(\mathbf{A}^*\mathbf{A}\right)_{j.}\left(\mathbf{a}_{i.}\right)\right)^\alpha_\alpha}{\sum\limits_{\beta\in I_{r,n}} \left|\mathbf{A}^*\mathbf{A}\right|^\alpha_\alpha} = \frac{\sum\limits_{\beta\in J_{r,m}\{i\}} \operatorname{cdet}_i\left(\left(\mathbf{A}\mathbf{A}^*\right)_{.i}\left(\mathbf{a}_{.j}\right)\right)^\beta_\beta}{\sum\limits_{\beta\in J_{r,m}} \left|\mathbf{A}\mathbf{A}^*\right|^\beta_\beta}.$$

Remark 2.9. Suppose $\mathbf{A} \in \mathbb{H}_r^{n\times m}$. By Lemma 2.6 and Remark 2.8, for the η-Hermitian adjoint matrix $\mathbf{A}^{\eta*} = (a_{ij}^{\eta*})$ and η-skew-Hermitian adjoint matrix $-\mathbf{A}^{\eta*} = (a_{ij}^{-\eta*})$, determinantal representations of their Moore–Penrose inverses $(\mathbf{A}^{\eta*})^{\dagger} = \left((a_{ij}^{\eta*})^{\dagger}\right) \in \mathbb{H}^{m\times n}$ and $(-\mathbf{A}^{\eta*})^{\dagger} = \left((a_{ij}^{-\eta*})^{\dagger}\right)$ are respectively

$$(a_{ij}^{\eta*})^{\dagger} = -\eta\overline{(a_{ji})^{\dagger}}\eta == -\eta\frac{\sum\limits_{\alpha\in I_{r,n}\{j\}} \operatorname{rdet}_j\left((\mathbf{A}^*\mathbf{A})_{j.}(\mathbf{a}_{i.})\right)_{\alpha}^{\alpha}}{\sum\limits_{\beta\in I_{r,n}} |\mathbf{A}^*\mathbf{A}|_{\alpha}^{\alpha}}\eta \tag{6}$$

$$= -\eta\frac{\sum\limits_{\beta\in J_{r,m}\{i\}} \operatorname{cdet}_i\left((\mathbf{A}\mathbf{A}^*)_{.i}(\mathbf{a}_{.j})\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,m}} |\mathbf{A}\mathbf{A}^*|_{\beta}^{\beta}}\eta, \tag{7}$$

$$(a_{ij}^{-\eta*})^{\dagger} = \eta\overline{(a_{ji})^{\dagger}}\eta =$$

$$\eta\frac{\sum\limits_{\alpha\in I_{r,n}\{j\}} \operatorname{rdet}_j\left((\mathbf{A}^*\mathbf{A})_{j.}(\mathbf{a}_{i.})\right)_{\alpha}^{\alpha}}{\sum\limits_{\beta\in I_{r,n}} |\mathbf{A}^*\mathbf{A}|_{\alpha}^{\alpha}}\eta = \eta\frac{\sum\limits_{\beta\in J_{r,m}\{i\}} \operatorname{cdet}_i\left((\mathbf{A}\mathbf{A}^*)_{.i}(\mathbf{a}_{.j})\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,m}} |\mathbf{A}\mathbf{A}^*|_{\beta}^{\beta}}\eta.$$

Since the projection matrices $\mathbf{A}^{\dagger}\mathbf{A} =: \mathbf{Q}_A = (q_{ij})$ and $\mathbf{A}\mathbf{A}^{\dagger} =: \mathbf{P}_A = (p_{ij})$ are Hermitian, then $q_{ij} = \overline{q_{ji}}$ and $p_{ij} = \overline{p_{ji}}$ for all $i \neq j$. From Theorem 2.6 and Remark 2.8, it follows evidently the corollaries.

Corollary 2.2. *If* $\mathbf{A} \in \mathbb{H}_r^{m\times n}$, *then its inducted projection matrices* $\mathbf{Q}_A = (q_{ij})_{n\times n}$ *and* $\mathbf{P}_A = (p_{ij})_{m\times m}$ *are determined as follows*

$$q_{ij} = \frac{\sum\limits_{\beta\in J_{r,n}\{i\}} \operatorname{cdet}_i\left((\mathbf{A}^*\mathbf{A})_{.i}(\dot{\mathbf{a}}_{.j})\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r,n}} |\mathbf{A}^*\mathbf{A}|_{\beta}^{\beta}} = \frac{\sum\limits_{\alpha\in I_{r,n}\{j\}} \operatorname{rdet}_j\left((\mathbf{A}^*\mathbf{A})_{j.}(\dot{\mathbf{a}}_{i.})\right)_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{r,n}} |\mathbf{A}^*\mathbf{A}|_{\alpha}^{\alpha}}, \tag{8}$$

$$p_{ij} = \frac{\sum\limits_{\alpha\in I_{r,m}\{j\}} \operatorname{rdet}_j\left((\mathbf{A}\mathbf{A}^*)_{j.}(\ddot{\mathbf{a}}_{i.})\right)_{\alpha}^{\alpha}}{\sum\limits_{\alpha\in I_{r,m}} |\mathbf{A}\mathbf{A}^*|_{\alpha}^{\alpha}} = \frac{\sum\limits_{\beta\in J_{r,m}\{i\}} \operatorname{cdet}_i\left((\mathbf{A}\mathbf{A}^*)_{.i}(\ddot{\mathbf{a}}_{.j})\right)_{\beta}^{\beta}}{\sum\limits_{\alpha\in J_{r,m}} |\mathbf{A}\mathbf{A}^*|_{\beta}^{\beta}}, \tag{9}$$

where $\dot{\mathbf{a}}_{.j}$ *and* $\ddot{\mathbf{a}}_{i.}$, $\dot{\mathbf{a}}_{i.}$ *and* $\ddot{\mathbf{a}}_{.j}$ *are the jth columns and ith rows of* $\mathbf{A}^*\mathbf{A} \in \mathbb{H}^{n\times n}$ *and* $\mathbf{A}\mathbf{A}^* \in \mathbb{H}^{m\times m}$, *respectively.*

Cramer's rule for the system (2)

The next lemma gives the explicit matrix form of a general solution to the system (1).

Lemma 3.1. [21] *Suppose that* $\mathbf{A}_1 \in \mathbb{H}^{m\times n}$, $\mathbf{B}_1 \in \mathbb{H}^{r\times s}$, $\mathbf{C}_1 \in \mathbb{H}^{m\times s}$, $\mathbf{A}_2 \in \mathbb{H}^{k\times n}$, $\mathbf{B}_2 \in \mathbb{H}^{r\times p}$, $\mathbf{C}_2 \in \mathbb{H}^{k\times p}$ *are known and* $X \in \mathbb{H}^{n\times r}$ *is unknown. Put* $\mathrm{H} = \mathbf{A}_2\mathbf{L}_{A_1}$, $\mathbf{N} = \mathbf{R}_{B_1}\mathbf{B}_2$, $\mathbf{T} = \mathbf{R}_H\mathbf{A}_2$, $\mathbf{F} = \mathbf{B}_2\mathbf{L}_N$. *Then the system (1) is consistent if and only if*

$$\mathbf{A}_i\mathbf{A}_i^{\dagger}\mathbf{C}_i\mathbf{B}_i^{\dagger}\mathbf{B}_i = \mathbf{C}_i, \quad i = 1,2; \quad \mathbf{T}\left[\mathbf{A}_2^{\dagger}\mathbf{X}\mathbf{B}_2^{\dagger} - \mathbf{A}_1^{\dagger}\mathbf{C}_1\mathbf{B}_1^{\dagger}\right]\mathbf{F} = 0.$$

In that case, the general solution of (1) can be expressed as

$$\begin{aligned}\mathbf{X} = {} & \mathbf{A}_1^{\dagger}\mathbf{C}_1\mathbf{B}_1^{\dagger} + \mathbf{L}_{A_1}\mathbf{H}^{\dagger}\mathbf{A}_2\mathbf{L}_T\left(\mathbf{A}_2^{\dagger}\mathbf{C}_2\mathbf{B}_2^{\dagger} - \mathbf{A}_1^{\dagger}\mathbf{C}_1\mathbf{B}_1^{\dagger}\right)\mathbf{B}_2\mathbf{B}_2^{\dagger} + \mathbf{T}^{\dagger}\mathbf{T}\left(\mathbf{A}_2^{\dagger}\mathbf{C}_2\mathbf{B}_2^{\dagger} - \mathbf{A}_1^{\dagger}\mathbf{C}_1\mathbf{B}_1^{\dagger}\right)\mathbf{B}_2\mathbf{N}^{\dagger}\mathbf{R}_{B_1} \\ & + \mathbf{L}_{A_1}\left(\mathbf{Z} - \mathbf{H}^{\dagger}\mathbf{H}\mathbf{Z}\mathbf{B}_2\mathbf{B}_2^{\dagger}\right) - \mathbf{L}_{A_1}\mathbf{H}^{\dagger}\mathbf{A}_2\mathbf{L}_T\mathbf{W}\mathbf{N}\mathbf{B}_2^{\dagger} + \left(\mathbf{W} - \mathbf{T}^{\dagger}\mathbf{T}\mathbf{W}\mathbf{N}\mathbf{N}^{\dagger}\right)\mathbf{R}_{B_1}\end{aligned} \tag{10}$$

where $\mathbf{Z}$ *and* $\mathbf{W}$ *are arbitrary matrices over* $\mathbb{H}$ *with appropriate sizes.*

Some simplification of (10) can be derived due to Lemma 2.2. So, we have,

$$\begin{aligned}
&\mathbf{L}_{A_1}\mathbf{H}^{\dagger}=\mathbf{L}_{A_1}(\mathbf{A}_2\mathbf{L}_{A_1})^{\dagger}=(\mathbf{A}_2\mathbf{L}_{A_1})^{\dagger}=\mathbf{H}^{\dagger},\\
&\mathbf{N}^{\dagger}\mathbf{R}_{B_1}=(\mathbf{R}_{B_1}\mathbf{B}_2)^{\dagger}\mathbf{R}_{B_1}=(\mathbf{R}_{B_1}\mathbf{B}_2)^{\dagger}=\mathbf{N}^{\dagger},\\
&\mathbf{T}^{\dagger}\mathbf{T}=(\mathbf{R}_H\mathbf{A}_2)^{\dagger}\mathbf{R}_H\mathbf{A}_2=(\mathbf{R}_H\mathbf{A}_2)^{\dagger}\mathbf{A}_2=\mathbf{T}^{\dagger}\mathbf{A}_2,\\
&\mathbf{L}_T=\mathbf{I}-\mathbf{T}^{\dagger}\mathbf{T}=\mathbf{I}-\mathbf{T}^{\dagger}\mathbf{A}_2.
\end{aligned}\tag{11}$$

Substituting (11) in (10), we get

$$\begin{aligned}
\mathbf{X}=&\ \mathbf{A}_1^{\dagger}\mathbf{C}_1\mathbf{B}_1^{\dagger}+\mathbf{H}^{\dagger}\mathbf{A}_2\big(\mathbf{I}-\mathbf{T}^{\dagger}\mathbf{A}_2\big)\big(\mathbf{A}_2^{\dagger}\mathbf{C}_2\mathbf{B}_2^{\dagger}-\mathbf{A}_1^{\dagger}\mathbf{C}_1\mathbf{B}_1^{\dagger}\big)\mathbf{B}_2\mathbf{B}_2^{\dagger}\\
&+\mathbf{T}^{\dagger}\mathbf{A}_2\big(\mathbf{A}_2^{\dagger}\mathbf{C}_2\mathbf{B}_2^{\dagger}-\mathbf{A}_1^{\dagger}\mathbf{C}_1\mathbf{B}_1^{\dagger}\big)\ \mathbf{B}_2\mathbf{N}^{\dagger}+\mathbf{L}_{A_1}\big(\mathbf{Z}-\mathbf{H}^{\dagger}\mathbf{H}\mathbf{Z}\mathbf{B}_2\mathbf{B}_2^{\dagger}\big)\\
&-\mathbf{H}^{\dagger}\mathbf{A}_2\mathbf{L}_T\mathbf{W}\mathbf{N}\mathbf{B}_2^{\dagger}+\big(\mathbf{W}-\mathbf{T}^{\dagger}\mathbf{T}\mathbf{W}\mathbf{N}\mathbf{N}^{\dagger}\big)\mathbf{R}_{B_1}\\
=&\ \mathbf{A}_1^{\dagger}\mathbf{C}_1\mathbf{B}_1^{\dagger}+\mathbf{H}^{\dagger}\mathbf{C}_2\mathbf{B}_2^{\dagger}+\mathbf{H}^{\dagger}\big(\mathbf{A}_2\mathbf{T}^{\dagger}-\mathbf{I}\big)\mathbf{A}_2\mathbf{A}_1^{\dagger}\mathbf{C}_1\mathbf{B}_1^{\dagger}\mathbf{P}_{B_2}\\
&-\mathbf{H}^{\dagger}\mathbf{A}_2\mathbf{T}^{\dagger}\mathbf{C}_2\mathbf{B}_2^{\dagger}+\mathbf{T}^{\dagger}\mathbf{C}_2\mathbf{N}^{\dagger}-\mathbf{T}^{\dagger}\mathbf{A}_2\mathbf{A}_1^{\dagger}\mathbf{C}_1\mathbf{B}_1^{\dagger}\mathbf{B}_2\mathbf{N}^{\dagger}\\
&+\mathbf{L}_{A_1}\big(\mathbf{Z}-\mathbf{H}^{\dagger}\mathbf{H}\mathbf{Z}\mathbf{B}_2\mathbf{B}_2^{\dagger}\big)-\mathbf{H}^{\dagger}\mathbf{A}_2\mathbf{L}_T\mathbf{W}\mathbf{N}\mathbf{B}_2^{\dagger}+\big(\mathbf{W}-\mathbf{T}^{\dagger}\mathbf{T}\mathbf{W}\mathbf{N}\mathbf{N}^{\dagger}\big)\mathbf{R}_{B_1}.
\end{aligned}$$

By putting $\mathbf{Z}=\mathbf{W}=\mathbf{0}$, we get the following expression of the partial solution

$$\begin{aligned}
\mathbf{X}_0=&\ \mathbf{A}_1^{\dagger}\mathbf{C}_1\mathbf{B}_1^{\dagger}+\mathbf{H}^{\dagger}\mathbf{C}_2\mathbf{B}_2^{\dagger}+\mathbf{T}^{\dagger}\mathbf{C}_2\mathbf{N}^{\dagger}+\mathbf{H}^{\dagger}\mathbf{A}_2\mathbf{T}^{\dagger}\mathbf{A}_2\mathbf{A}_1^{\dagger}\mathbf{C}_1\mathbf{B}_1^{\dagger}\mathbf{P}_{B_2}\\
&-\mathbf{H}^{\dagger}\mathbf{A}_2\mathbf{A}_1^{\dagger}\mathbf{C}_1\mathbf{B}_1^{\dagger}\mathbf{P}_{B_2}-\mathbf{H}^{\dagger}\mathbf{A}_2\mathbf{T}^{\dagger}\mathbf{C}_2\mathbf{B}_2^{\dagger}-\mathbf{T}^{\dagger}\mathbf{A}_2\mathbf{A}_1^{\dagger}\mathbf{C}_1\mathbf{B}_1^{\dagger}\mathbf{B}_2\mathbf{N}^{\dagger}.
\end{aligned}\tag{12}$$

Now consider the system (2). We have

$$\mathbf{Q}_{A_i^{\eta*}}=\big(\mathbf{A}_i^{\eta*}\big)^{\dagger}\mathbf{A}_i^{\eta*}=\big(\mathbf{A}_i\mathbf{A}_i^{\dagger}\big)^{\eta*}=\mathbf{P}_{A_i}^{\eta},$$

similarly, $\mathbf{P}_{A_i^{\eta*}}=\mathbf{Q}_{A_i}^{\eta}$, and, by Lemma 2.1, $\mathbf{L}_{A_i^{\eta*}}=\mathbf{R}_{A_i}^{\eta}$, and $\mathbf{R}_{A_i^{\eta*}}=\mathbf{L}_{A_i}^{\eta}$ for $i=1,2$. Moreover, by substituting $\mathbf{B}_i=\mathbf{A}_i^{\eta*}$, we obtain

$$\begin{aligned}
&\mathbf{N}=\mathbf{R}_{A_1^{\eta*}}\mathbf{A}_2^{\eta*}=(\mathbf{A}_2\mathbf{L}_{A_1})^{\eta*}=\mathbf{H}^{\eta*},\\
&\mathbf{F}=\mathbf{A}_2^{\eta*}\mathbf{L}_{H^{\eta*}}=(\mathbf{R}_H\mathbf{A}_2)^{\eta*}=\mathbf{T}^{\eta*}.
\end{aligned}$$

From above, it follows the next analog of Lemma 3.1.

Lemma 3.2. *Suppose that* $\mathbf{A}_1\in\mathbb{H}^{m\times n}$, $\mathbf{A}_2\in\mathbb{H}^{k\times n}$, $\mathbf{C}_1\in\mathbb{H}^{m\times m}$, $\mathbf{C}_2\in\mathbb{H}^{k\times k}$ *are known and* $\mathbf{X}\in\mathbb{H}^{n\times n}$ *is unknown. The system (2) is consistent if and only if*

$$\mathbf{P}_{A_i}\mathbf{C}_i\mathbf{P}_{A_i}^{\eta}=\mathbf{C}_i,\quad i=1,2;\tag{13}$$

$$\mathbf{T}\Big[\mathbf{A}_2^{\dagger}\mathbf{C}_2(\mathbf{A}_2^{\eta*})^{\dagger}-\mathbf{A}_1^{\dagger}\mathbf{C}_1(\mathbf{A}_1^{\eta*})^{\dagger}\Big]\mathbf{T}^{\eta*}=0.\tag{14}$$

In that case, the general solution to (2) is expressed as

$$\begin{aligned}
\mathbf{X}=&\ \mathbf{A}_1^{\dagger}\mathbf{C}_1\big(\mathbf{A}_1^{\eta*}\big)^{\dagger}+\mathbf{H}^{\dagger}\mathbf{C}_2(\mathbf{A}_2^{\eta*})^{\dagger}+\mathbf{H}^{\dagger}\big(\mathbf{A}_2\mathbf{T}^{\dagger}-\mathbf{I}\big)\mathbf{A}_2\mathbf{A}_1^{\dagger}\mathbf{C}_1(\mathbf{A}_1^{\eta*})^{\dagger}\mathbf{Q}_{A_2}-\mathbf{H}^{\dagger}\mathbf{A}_2\mathbf{T}^{\dagger}\mathbf{C}_2(\mathbf{A}_2^{\eta*})^{\dagger}+\mathbf{T}^{\dagger}\mathbf{C}_2(\mathbf{H}^{\eta*})^{\dagger}\\
&-\mathbf{T}^{\dagger}\mathbf{A}_2\mathbf{A}_1^{\dagger}\mathbf{C}_1(\mathbf{A}_1^{\eta*})^{\dagger}\mathbf{A}_2^{\eta*}(\mathbf{H}^{\eta*})^{\dagger}+\mathbf{L}_{A_1}\Big(\mathbf{Z}-\mathbf{H}^{\dagger}\mathbf{H}\mathbf{Z}\mathbf{A}_2^{\eta*}(\mathbf{A}_2^{\eta*})^{\dagger}\Big)-\mathbf{H}^{\dagger}\mathbf{A}_2\mathbf{L}_T\mathbf{W}\mathbf{H}^{\eta*}(\mathbf{A}_2^{\eta*})^{\dagger}\\
&+\Big(\mathbf{W}-\mathbf{T}^{\dagger}\mathbf{T}\mathbf{W}\mathbf{H}^{\eta*}(\mathbf{H}^{\eta*})^{\dagger}\Big)\mathbf{L}_{A_1}.
\end{aligned}$$

where $\mathbf{Z}$ *and* $\mathbf{W}$ *are arbitrary matrices over* $\mathbb{H}$ *with appropriate sizes.*

By putting $\mathbf{Z}$, $\mathbf{W}$ as zero-matrices, the partial solution to (2) is

$$\begin{aligned}
\mathbf{X}=&\ \mathbf{A}_1^{\dagger}\mathbf{C}_1\big(\mathbf{A}_1^{\eta*}\big)^{\dagger}+\mathbf{H}^{\dagger}\mathbf{C}_2(\mathbf{A}_2^{\eta*})^{\dagger}+\mathbf{T}^{\dagger}\mathbf{C}_2(\mathbf{H}^{\eta*})^{\dagger}+\mathbf{H}^{\dagger}\mathbf{A}_2\mathbf{T}^{\dagger}\mathbf{A}_2\mathbf{A}_1^{\dagger}\mathbf{C}_1(\mathbf{A}_1^{\eta*})^{\dagger}\mathbf{Q}_{A_2}-\mathbf{H}^{\dagger}\mathbf{A}_2\mathbf{A}_1^{\dagger}\mathbf{C}_1(\mathbf{A}_1^{\eta*})^{\dagger}\mathbf{Q}_{A_2}\\
&-\mathbf{H}^{\dagger}\mathbf{A}_2\mathbf{T}^{\dagger}\mathbf{C}_2(\mathbf{A}_2^{\eta*})^{\dagger}-\mathbf{T}^{\dagger}\mathbf{A}_2\mathbf{A}_1^{\dagger}\mathbf{C}_1(\mathbf{A}_1^{\eta*})^{\dagger}\mathbf{A}_2^{\eta*}(\mathbf{H}^{\eta*})^{\dagger}.
\end{aligned}\tag{15}$$

Further, we give determinantal representations of (15).

Suppose that $\mathbf{A}_1 \in \mathbb{H}^{m\times n}_{r_1}$, $\mathbf{A}_2 \in \mathbb{H}^{k\times n}_{r_2}$, $\mathbf{C}_1 = \left(c^{(1)}_{ij}\right) \in \mathbb{H}^{m\times m}$, $\mathbf{C}_2 = \left(c^{(2)}_{ij}\right) \in \mathbb{H}^{k\times k}$, rank$\mathbf{H} = r_3$, and rank$\mathbf{T} = r_4$. So, $\mathbf{A}_1^{\dagger} = \left(a^{(1),\dagger}_{ij}\right) \in \mathbb{H}^{n\times m}$, $(\mathbf{A}_1^{\eta *})^{\dagger} = \left(a^{(1),\eta *,\dagger}_{ij}\right) \in \mathbb{H}^{m\times n}$, $\mathbf{A}_2^{\dagger} = \left(a^{(2),\dagger}_{ij}\right) \in \mathbb{H}^{k\times n}$, $(\mathbf{A}_2^{\eta *})^{\dagger} = \left(a^{(2),\eta *,\dagger}_{ij}\right) \in \mathbb{H}^{n\times k}$, $\mathbf{H}^{\dagger} = \left(h^{\dagger}_{ij}\right) \in \mathbb{H}^{n\times k}$, and $\mathbf{T}^{\dagger} = \left(t^{\dagger}_{ij}\right) \in \mathbb{H}^{n\times k}$.

Consider each summand of (15) separately.

(i) Denote $\mathbf{C}_{11} := \mathbf{A}_1^{*}\mathbf{C}_1\mathbf{A}_1^{\eta}$. For the first term of (15) $\mathbf{X}_1 = \mathbf{A}_1^{\dagger}\mathbf{C}_1\left(\mathbf{A}_1^{\eta *}\right)^{\dagger} = (x^{(1)}_{ij})$, we have

$$x^{(1)}_{ij} = \sum_{l=1}^{m}\sum_{t=1}^{m} a^{(1),\dagger}_{il} c^{(1)}_{lt} a^{(1),\eta *,\dagger}_{tj}.$$

Taking into account (4) and (6) for $\mathbf{A}_1^{\dagger}$ and $\left(\mathbf{A}_1^{\eta *}\right)^{\dagger}$, respectively, we get

$$x^{(1)}_{ij} = \frac{\sum\limits_{l=1}^{m}\sum\limits_{t=1}^{m}\sum\limits_{\beta\in J_{r_1,n}\{i\}} \mathrm{cdet}_i\left(\left(\mathbf{A}_1^{*}\mathbf{A}_1\right)_{.i}\left(\mathbf{a}^{(1),*}_{.l}\right)\right)_{\beta}^{\beta} c^{(1)}_{lt}\left(-\eta\sum\limits_{\alpha\in I_{r_1,n}\{j\}}\mathrm{rdet}_j\left(\left(\mathbf{A}_1^{*}\mathbf{A}_1\right)_{j.}\left(\mathbf{a}^{(1)}_{t.}\right)\right)_{\alpha}^{\alpha}\eta\right)}{\sum\limits_{\alpha\in I_{r_1,n}}\left|\mathbf{A}_1^{*}\mathbf{A}_1\right|_{\alpha}^{\alpha}\sum\limits_{\beta\in J_{r_1,n}}\left|\mathbf{A}_1^{*}\mathbf{A}_1\right|_{\beta}^{\beta}}.$$

Suppose that $e_{l.}$ and $e_{.l}$ are the unit row and column vectors such that all their components are 0 except the lth components which are 1.

Since $\sum\limits_{l=1}^{m}\sum\limits_{t=1}^{m} a^{(1),*}_{fl} c^{(1)}_{lt} a^{(1),\eta}_{ts} = c^{(11)}_{fs}$, then

$$x^{(1)}_{ij} = \frac{\sum\limits_{f=1}^{n}\sum\limits_{s=1}^{n}\sum\limits_{\beta\in J_{r_1,n}\{i\}} \mathrm{cdet}_i\left(\left(\mathbf{A}_1^{*}\mathbf{A}_1\right)_{.i}\left(\mathbf{e}_{.f}\right)\right)_{\beta}^{\beta} c^{(11)}_{fs}\left(-\eta\sum\limits_{\alpha\in I_{r_1,n}\{j\}}\mathrm{rdet}_j\left(\left(\mathbf{A}_1^{*}\mathbf{A}_1\right)_{j.}\left(\mathbf{e}_{s.}\right)\right)_{\alpha}^{\alpha}\eta\right)}{\sum\limits_{\alpha\in I_{r_1,n}}\left|\mathbf{A}_1^{*}\mathbf{A}_1\right|_{\alpha}^{\alpha}\sum\limits_{\beta\in J_{r_1,n}}\left|\mathbf{A}_1^{*}\mathbf{A}_1\right|_{\beta}^{\beta}}. \tag{16}$$

By

$$v^{(1)}_{is} := \sum_{f=1}^{n}\sum_{\beta\in J_{r_1,n}\{i\}} \mathrm{cdet}_i\left(\left(\mathbf{A}_1^{*}\mathbf{A}_1\right)_{.i}\left(\mathbf{e}_{.f}\right)\right)_{\beta}^{\beta} c^{(11)}_{fs} = \sum_{\beta\in J_{r_1,n}\{i\}} \mathrm{cdet}_i\left(\left(\mathbf{A}_1^{*}\mathbf{A}_1\right)_{.i}\left(\mathbf{c}^{(11)}_{.s}\right)\right)_{\beta}^{\beta}, \tag{17}$$

denote the sth component of a row-vector $v^{(1)}_{i.} = \left[v^{(1)}_{i1}, \dots, v^{(1)}_{in}\right]$. Then

$$\sum_{s=1}^{m} v^{(1)}_{is}\left(-\eta\sum_{\alpha\in I_{r_1,n}\{j\}}\mathrm{rdet}_j\left(\left(\mathbf{A}_1^{*}\mathbf{A}_1\right)_{j.}\left(\mathbf{e}_{s.}\right)\right)_{\alpha}^{\alpha}\eta\right) = -\eta\sum_{\alpha\in I_{r_1,n}\{j\}}\mathrm{rdet}_j\left(\left(\mathbf{A}_1^{*}\mathbf{A}_1\right)_{j.}\left(\mathbf{v}^{(1),\eta}_{i.}\right)\right)_{\alpha}^{\alpha}\eta. \tag{18}$$

Farther, it's evident that $\sum\limits_{\beta\in J_{r_1,n}}\left|\mathbf{A}_1^{*}\mathbf{A}_1\right|_{\beta}^{\beta} = \sum\limits_{\alpha\in I_{r_1,n}}\left|\mathbf{A}_1^{*}\mathbf{A}_1\right|_{\alpha}^{\alpha}$. Integrating (17) and (18) in (16), the determinantal representation of the first term of (15) can be expressed as

$$x^{(1)}_{ij} = \frac{-\eta\sum\limits_{\alpha\in I_{r_1,n}\{j\}}\mathrm{rdet}_j\left(\left(\mathbf{A}_1^{*}\mathbf{A}_1\right)_{j.}\left(\mathbf{v}^{(1),\eta}_{i.}\right)\right)_{\alpha}^{\alpha}\eta}{\left(\sum\limits_{\alpha\in I_{r_1,n}}\left|\mathbf{A}_1^{*}\mathbf{A}_1\right|_{\alpha}^{\alpha}\right)^{2}} \tag{19}$$

where

$$\mathbf{v}^{(1),\eta}_{i.} = \left[-\eta\sum_{\beta\in J_{r_1,n}\{i\}}\mathrm{cdet}_i\left(\left(\mathbf{A}_1^{*}\mathbf{A}_1\right)_{.i}\left(\mathbf{c}^{(11)}_{.s}\right)\right)_{\beta}^{\beta}\eta\right] \in \mathbb{H}^{1\times n}, \quad s = 1, \dots, n. \tag{20}$$

If we denote by

$$v_{fj}^{(2)} := \sum_{s=1}^{n} c_{fs}^{(11)} \left(-\eta \sum_{\alpha \in I_{r_1,n}\{j\}} \mathrm{rdet}_j \left((\mathbf{A}_1^*\mathbf{A}_1)_{j.}(\mathbf{e}_{s.}) \right)_{\alpha}^{\alpha} \eta \right) = -\eta \sum_{\alpha \in I_{r_1,n}\{j\}} \mathrm{rdet}_j \left((\mathbf{A}_1^*\mathbf{A}_1)_{j.}(\mathbf{c}_{f.}^{(11),\eta}) \right)_{\alpha}^{\alpha} \eta, \tag{21}$$

the fth component of a column-vector $\mathbf{v}_{.j}^{(2)} = \left[v_{1j}^{(2)}, \dots, v_{nj}^{(2)} \right]$, then

$$\sum_{f=1}^{n} \sum_{\beta \in J_{r_1,n}\{i\}} \mathrm{cdet}_i \left(\left(\mathbf{A}_1^*\mathbf{A}_1 \right)_{.i} (\mathbf{e}_{.f}) \right)_{\beta}^{\beta} \mathbf{v}_{fj}^{(2)} = \sum_{\beta \in J_{r_1,n}\{i\}} \mathrm{cdet}_i \left(\left(\mathbf{A}_1^*\mathbf{A}_1 \right)_{.i} \left(\mathbf{v}_{.j}^{(2)} \right) \right)_{\beta}^{\beta}. \tag{22}$$

Integrating (21) and (22) in (16), we obtain another determinantal representation of the first term

$$x_{ij}^{(1)} = \frac{\sum_{\beta \in J_{r_1,n}\{i\}} \mathrm{cdet}_i \left(\left(\mathbf{A}_1^*\mathbf{A}_1 \right)_{.i} \left(\mathbf{v}_{.j}^{(2)} \right) \right)_{\beta}^{\beta}}{\left(\sum_{\beta \in J_{r_1,n}} \left| \mathbf{A}_1^*\mathbf{A}_1 \right|_{\beta}^{\beta} \right)^2}, \tag{23}$$

where

$$\mathbf{v}_{.j}^{(2)} = \left[-\eta \sum_{\alpha \in I_{r_1,n}\{j\}} \mathrm{rdet}_j \left((\mathbf{A}_1^*\mathbf{A}_1)_{j.}(\mathbf{c}_{f.}^{(11),\eta}) \right)_{\alpha}^{\alpha} \eta \right] \in \mathbb{H}^{n\times 1}, \quad f = 1, \dots, n,$$

are the column vector and $\mathbf{c}_{f.}^{(11),\eta}$ is the fth row of $\mathbf{C}_{11}^{\eta} = \mathbf{A}_1^{\eta *}\mathbf{C}_1^{\eta}\mathbf{A}_1$.

(ii) Similarly above, for the second term $\mathbf{X}_2 = \mathbf{H}^{\dagger}\mathbf{C}_2(\mathbf{A}_2^{\eta *})^{\dagger} = \left(x_{ij}^{(2)} \right)$ of (15), we have

$$x_{ij}^{(2)} = \frac{\sum_{\beta \in J_{r_3,n}\{i\}} \mathrm{cdet}_i \left((\mathbf{H}^*\mathbf{H})_{.i} \left(\mathbf{d}_{.j}^{A_2} \right) \right)_{\beta}^{\beta}}{\sum_{\beta \in J_{r_3,n}} |\mathbf{H}^*\mathbf{H}|_{\beta}^{\beta} \sum_{\alpha \in I_{r_2,n}} \left| \mathbf{A}_2^*\mathbf{A}_2 \right|_{\alpha}^{\alpha}}, \tag{24}$$

or

$$x_{ij}^{(2)} = \frac{-\eta \sum_{\alpha \in I_{r_2,n}\{j\}} \mathrm{rdet}_j \left(\left(\mathbf{A}_2^*\mathbf{A}_2 \right)_{j.} \left(\mathbf{d}_{i.}^{H} \right) \right)_{\alpha}^{\alpha} \eta}{\sum_{\beta \in J_{r_3,n}} |\mathbf{H}^*\mathbf{H}|_{\beta}^{\beta} \sum_{\alpha \in I_{r_2,n}} \left| \mathbf{A}_2^*\mathbf{A}_2 \right|_{\alpha}^{\alpha}}, \tag{25}$$

where

$$\mathbf{d}_{.j}^{A_2} = \left[-\eta \sum_{\alpha \in I_{r_2,n}\{j\}} \mathrm{rdet}_j \left(\left(\mathbf{A}_2^*\mathbf{A}_2 \right)_{j.} \left(\mathbf{c}_{q.}^{(21),\eta} \right) \right)_{\alpha}^{\alpha} \eta \right] \in \mathbb{H}^{n\times 1}, \quad q = 1, \dots, n,$$

$$\mathbf{d}_{i.}^{H} = \left[-\eta \sum_{\beta \in J_{r_3,n}\{i\}} \mathrm{cdet}_i \left((\mathbf{H}^*\mathbf{H})_{.i} \left(\mathbf{c}_{.l}^{(21)} \right) \right)_{\beta}^{\beta} \eta \right] \in \mathbb{H}^{1\times n}, \quad l = 1, \dots, n\,.$$

Here $\mathbf{c}_{q.}^{(21)}$ and $\mathbf{c}_{.l}^{(21)}$ are the qth row and the lth column of $\mathbf{C}_{21} = \mathbf{H}^*\mathbf{C}_2\mathbf{A}_2^{\eta}$.

(iii) The third term $\mathbf{X}_3 = \mathbf{T}^{\dagger}\mathbf{C}_2(\mathbf{H}^{\eta *})^{\dagger} = \left(x_{ij}^{(3)} \right)$ of (15) can be obtained similarly as well. So,

$$x_{ij}^{(3)} = \frac{\sum_{\beta \in J_{r_4,n}\{i\}} \mathrm{cdet}_i \left((\mathbf{T}^*\mathbf{T})_{.i} \left(\mathbf{d}_{.j}^{H} \right) \right)_{\beta}^{\beta}}{\sum_{\beta \in J_{r_4,n}} |\mathbf{T}^*\mathbf{T}|_{\beta}^{\beta} \sum_{\alpha \in I_{r_3,n}} |\mathbf{H}^*\mathbf{H}|_{\alpha}^{\alpha}}, \tag{26}$$

or

$$x_{ij}^{(3)} = \frac{-\eta \sum\limits_{\alpha \in I_{r_3,r}\{j\}} \mathrm{rdet}_j\left((\mathbf{H}^*\mathbf{H})_{j.}\left(\mathbf{d}_{i.}^{T}\right)\right)_{\alpha}^{\alpha}\eta}{\sum\limits_{\beta \in J_{r_4,n}} |\mathbf{T}^*\mathbf{T}|_{\beta}^{\beta} \sum\limits_{\alpha \in I_{r_3,n}} |\mathbf{H}^*\mathbf{H}|_{\alpha}^{\alpha}}, \tag{27}$$

where

$$\mathbf{d}_{.j}^{H} = \left[-\eta \sum_{\alpha \in I_{r_3,n}\{f\}} \mathrm{rdet}_j\left((\mathbf{H}^*\mathbf{H})_{j.}\left(\mathbf{c}_{q.}^{(22),\eta}\right)\right)_{\alpha}^{\alpha}\eta\right] \in \mathbb{H}^{n\times 1}, \quad q = 1,\ldots,n,$$
$$\mathbf{d}_{i.}^{T} = \left[-\eta \sum_{\beta \in J_{r_4,n}\{i\}} \mathrm{cdet}_i\left((\mathbf{T}^*\mathbf{T})_{.i}\left(\mathbf{c}_{.l}^{(22)}\right)\right)_{\beta}^{\beta}\eta\right] \in \mathbb{H}^{1\times n}, \quad l = 1,\ldots,n.$$

Here $c_{q.}^{(22)}$, $c_{.l}^{(22)}$ are the qth row and the lth column of $\mathbf{C}_{22} = \mathbf{T}^*\mathbf{C}_2\mathbf{H}^{\eta}$.

(iv) Now, consider the fourth term $\mathbf{X}_4 = \mathbf{H}^{\dagger}\mathbf{A}_2\mathbf{T}^{\dagger}\mathbf{A}_2\mathbf{A}_1^{\dagger}\mathbf{C}_1(\mathbf{A}_1^{\eta *})^{\dagger}\mathbf{Q}_{A_2} = \left(x_{ij}^{(4)}\right)$ of (15). Taking into account (4) for determinantal representations of $\mathbf{H}^{\dagger}$ and $\mathbf{T}^{\dagger}$, we get

$$x_{ij}^{(4)} = \frac{\sum\limits_{s=1}^{n}\sum\limits_{z=1}^{n}\sum\limits_{f=1}^{n} \sum\limits_{\beta \in J_{r_3,n}\{i\}} \mathrm{cdet}_i\left((\mathbf{H}^*\mathbf{H})_{.i}\left(\mathbf{a}_{.s}^{(2,H)}\right)\right)_{\beta}^{\beta} \sum\limits_{\beta \in J_{r_4,n}\{s\}} \mathrm{cdet}_s\left((\mathbf{T}^*\mathbf{T})_{.s}\left(\mathbf{a}_{.z}^{(2,T)}\right)\right)_{\beta}^{\beta} x_{zf}^{(1)} q_{fj}}{\sum\limits_{\beta \in J_{r_3,n}} |\mathbf{H}^*\mathbf{H}|_{\beta}^{\beta} \sum\limits_{\beta \in J_{r_4,n}} |\mathbf{T}^*\mathbf{T}|_{\beta}^{\beta}}, \tag{28}$$

Here $\boldsymbol{a}_{.i}^{(2,H)}$, $\boldsymbol{a}_{.i}^{(2,T)}$ denote the ith columns of $\mathbf{H}^*\mathbf{A}_2$ and $\mathbf{T}^*\mathbf{A}_2$, respectively. $x_{zf}^{(1)}$ is the (zf)th element of the first term that is obtained in the point (i). q_{fj} is the (fj)th element of $\mathbf{Q}_{A_2}$ that, by (8), can be expessed as

$$q_{fj} = \frac{\sum\limits_{\alpha \in I_{r_2,n}\{j\}} \mathrm{rdet}_j\left(\left(\mathbf{A}_2^*\mathbf{A}_2\right)_{j.}\left(\dot{\mathbf{a}}_{f.}^{(2)}\right)\right)_{\alpha}^{\alpha}}{\sum\limits_{\alpha \in I_{r_2,n}} \left|\mathbf{A}_2^*\mathbf{A}_2\right|_{\alpha}^{\alpha}},$$

where $\dot{\mathbf{a}}_{f.}^{(2)}$ is the fth row of $\mathbf{A}_2^*\mathbf{A}_2$. Denote

$$q_{zj}^{(1)} := \sum_{f=1}^{n} x_{zf}^{(1)} \sum_{\alpha \in I_{r_2,n}\{j\}} \mathrm{rdet}_j\left(\left(\mathbf{A}_2^*\mathbf{A}_2\right)_{j.}\left(\dot{\mathbf{a}}_{f.}^{(2)}\right)\right)_{\alpha}^{\alpha} = \sum_{\alpha \in I_{r_2,n}\{j\}} \mathrm{rdet}_j\left(\left(\mathbf{A}_2^*\mathbf{A}_2\right)_{j.}\left(\widetilde{\mathbf{x}}_{z.}^{(1)}\right)\right)_{\alpha}^{\alpha} \tag{29}$$

where $\widetilde{\mathbf{x}}_{z.}^{(1)}$ is the zth row of $\widetilde{\mathbf{X}}_1 = \mathbf{X}_1\mathbf{A}_2^*\mathbf{A}_2$ for all z, $j = 1,\ldots, n$ and $\mathbf{X}_1$ is found in the point (i). Construct the matrix $\mathbf{Q}_1 = (q_{zj}^{(1)}) \in \mathbb{H}^{n\times n}$. Further, denote

$$t_{sj}^{(1)} := \sum_{z=1}^{n} \sum_{\beta \in J_{r_4,n}\{s\}} \mathrm{cdet}_s\left((\mathbf{T}^*\mathbf{T})_{.s}\left(\mathbf{a}_{.z}^{(2,T)}\right)\right)_{\beta}^{\beta} q_{zj}^{(1)} = \sum_{\beta \in J_{r_4,n}\{s\}} \mathrm{cdet}_s\left((\mathbf{T}^*\mathbf{T})_{.s}\left(\widetilde{\mathbf{t}}_{.j}\right)\right)_{\beta}^{\beta},$$

where $\widetilde{\mathbf{t}}_{.j}$ is the jth column of $\widetilde{\mathbf{T}} = \mathbf{T}^*\mathbf{A}_2\mathbf{Q}_1$ and construct the matrix $\mathbf{T}_1 = (t_{qj}^{(1)}) \in \mathbb{H}^{n\times n}$. Finally, denote $\widetilde{\mathbf{H}} := \mathbf{H}^*\mathbf{A}_2\mathbf{T}_1$. From these denotations and the equation (28), it follows

$$x_{ij}^{(4)} = \frac{\sum\limits_{\beta \in J_{r_3,n}\{i\}} \mathrm{cdet}_i\left((\mathbf{H}^*\mathbf{H})_{.i}\left(\widetilde{\mathbf{h}}_{.j}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta \in J_{r_3,n}} |\mathbf{H}^*\mathbf{H}|_{\beta}^{\beta} \sum\limits_{\beta \in J_{r_4,n}} |\mathbf{T}^*\mathbf{T}|_{\beta}^{\beta} \sum\limits_{\alpha \in I_{r_2,n}} \left|\mathbf{A}_2^*\mathbf{A}_2\right|_{\alpha}^{\alpha}}, \tag{30}$$

where $\widetilde{\mathbf{h}}_{.j}$ is the jth column of $\widetilde{\mathbf{H}}$.

(v) For $\mathbf{X}_5 = \mathbf{H}^{\dagger}\mathbf{A}_2\mathbf{A}_1^{\dagger}\mathbf{C}_1(\mathbf{A}_1^{\eta *})^{\dagger}\mathbf{Q}_{A_2} = \left(x_{ij}^{(5)}\right)$, we have

$$x_{ij}^{(5)} = \frac{\sum\limits_{s=1}^{n}\sum\limits_{f=1}^{n}\sum\limits_{\beta\in J_{r_3,n}\{i\}} \mathrm{cdet}_i\left((\mathbf{H}^*\mathbf{H})_{.i}\left(\mathbf{a}_{.s}^{(2,H)}\right)\right)_{\beta}^{\beta} x_{sf}^{(1)}\, q_{fj}}{\sum\limits_{\beta\in J_{r_3,n}} |\mathbf{H}^*\mathbf{H}|_{\beta}^{\beta}}.$$

Denote $\widehat{\mathbf{H}} := \mathbf{H}^*\mathbf{A}_2\mathbf{Q}_1$, where $\mathbf{Q}_1 = (q_{sj}^{(1)})$ is determined in (29). So, similarly to the previous case, we obtain

$$x_{ij}^{(5)} = \frac{\sum\limits_{\beta\in J_{r_3,n}\{i\}} \mathrm{cdet}_i\left((\mathbf{H}^*\mathbf{H})_{.i}\left(\widehat{\mathbf{h}}_{.j}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{r_3,n}} |\mathbf{H}^*\mathbf{H}|_{\beta}^{\beta} \sum\limits_{\alpha\in I_{r_2,n}} \left|\mathbf{A}_2^*\mathbf{A}_2\right|_{\alpha}^{\alpha}}, \tag{31}$$

where $\widehat{\mathbf{h}}_{.j}$ is the jth column of $\widehat{\mathbf{H}}$.

(vi) Consider the sixth term $\mathbf{X}_6 = \mathbf{H}^{\dagger}\mathbf{A}_2\mathbf{T}^{\dagger}\mathbf{C}_2(\mathbf{A}_2^{\eta *})^{\dagger} = \left(x_{ij}^{(6)}\right)$. So,

$$x_{ij}^{(6)} = \frac{\sum\limits_{q=1}^{n}\sum\limits_{\beta\in J_{r_3,n}\{i\}} \mathrm{cdet}_i\left((\mathbf{H}^*\mathbf{H})_{.i}\left(\mathbf{a}_{.q}^{(2,H)}\right)\right)_{\beta}^{\beta}\phi_{qj}}{\sum\limits_{\beta\in J_{r_3,n}} |\mathbf{H}^*\mathbf{H}|_{\beta}^{\beta}\sum\limits_{\beta\in J_{r_4,n}} |\mathbf{T}^*\mathbf{T}|_{\beta}^{\beta}\sum\limits_{\beta\in J_{r_2,n}} \left|\mathbf{A}_2^*\mathbf{A}_2\right|_{\beta}^{\beta}}, \tag{32}$$

where

$$\phi_{qj} = \sum_{\beta\in J_{r_4,n}\{q\}} \mathrm{cdet}_q\left((\mathbf{T}^*\mathbf{T})_{.q}\left(\varphi_{.j}^{A_2}\right)\right)_{\beta}^{\beta} = -\eta\sum_{\alpha\in I_{r_2,n}\{j\}} \mathrm{rdet}_j\left((\mathbf{A}_2^*\mathbf{A}_2)_{j.}\left(\varphi_{q.}^{T}\right)\right)_{\alpha}^{\alpha}\eta, \tag{33}$$

and

$$\varphi_{.j}^{A_2} = \left[-\eta\sum_{\alpha\in I_{r_2,n}\{f\}} \mathrm{rdet}_j\left((\mathbf{A}_2^*\mathbf{A}_2)_{j.}\left(\mathbf{c}_{q.}^{(23),\eta}\right)\right)_{\alpha}^{\alpha}\eta\right] \in \mathbb{H}^{1\times n}, \quad q = 1,\ldots,n,$$

$$\varphi_{q.}^{T} = \left[-\eta\sum_{\beta\in J_{r_4,n}\{q\}} \mathrm{cdet}_q\left((\mathbf{T}^*\mathbf{T})_{.q}\left(\mathbf{c}_{.l}^{(23)}\right)\right)_{\beta}^{\beta}\eta\right] \in \mathbb{H}^{n\times 1}, \quad l = 1,\ldots,n.$$

Here $\mathbf{c}_{.l}^{(23)}$ is the lth column of $\mathbf{C}_{23} = \mathbf{T}^*\mathbf{C}_2\mathbf{A}_2^{\eta}$ and $\mathbf{c}_{q.}^{(23),\eta}$ is the qth row of $\mathbf{C}_{23}^{\eta}$. Construct the matrix $\mathbf{\Phi} = (\phi_{qj})$ such that ϕ_{qj} is determined in (33) and denote $\widetilde{\mathbf{\Phi}} := \mathbf{H}^*\mathbf{A}_2\Phi$. From this denotation and the equation (32), it follows

$$x_{ij}^{(6)} = \frac{\sum\limits_{\beta\in J_{r_3,n}\{i\}} \mathrm{cdet}_i\left((\mathbf{H}^*\mathbf{H})_{.i}\left(\widetilde{\phi}_{.j}\right)\right)_{\beta}}{\sum\limits_{\beta\in J_{r_3,n}} |\mathbf{H}^*\mathbf{H}|_{\beta}^{\beta}\sum\limits_{\beta\in J_{r_4,n}} |\mathbf{T}^*\mathbf{T}|_{\beta}^{\beta}\sum\limits_{\beta\in J_{r_2,n}} \left|\mathbf{A}_2^*\mathbf{A}_2\right|_{\beta}^{\beta}}, \tag{34}$$

where $\widetilde{\phi}_{.j}$ is the jth column of $\widetilde{\mathbf{\Phi}}$.

(vii) Finally, consider the seventh term $\mathbf{X}_7 = \mathbf{T}^{\dagger}\mathbf{A}_2\mathbf{A}_1^{\dagger}\mathbf{C}_1(\mathbf{A}_1^{\eta *})^{\dagger}\mathbf{A}_2^{\eta *}(\mathbf{H}^{\eta *})^{\dagger} = \left(x_{ij}^{(7)}\right)$ of (15). Taking into account (4) for $T^{\dagger}$ and (6) for $(H^{\eta *})^{\dagger}$, we get

$$x_{ij}^{(7)} = \frac{\sum\limits_{q=1}^{n}\sum\limits_{f=1}^{n}\sum\limits_{\beta\in J_{r_4,n}\{i\}} \mathrm{cdet}_i\left((\mathbf{T}^*\mathbf{T})_{.i}\left(\mathbf{a}_{.q}^{(2,T)}\right)\right)_{\beta}^{\beta} x_{qf}^{(1)}\left(-\eta\sum\limits_{\alpha\in I_{r_3,n}\{j\}} \mathrm{rdet}_j\left((\mathbf{H}^*\mathbf{H})_{j.}\left(\mathbf{a}_{f.}^{(2,H,\eta *)}\right)\right)_{\alpha}^{\alpha}\eta\right)}{\sum\limits_{\beta\in J_{r_4,n}} |\mathbf{T}^*\mathbf{T}|_{\beta}^{\beta}\sum\limits_{\alpha\in I_{r_3,r}} |\mathbf{H}^*\mathbf{H}|_{\alpha}^{\alpha}}, \tag{35}$$

where $\mathbf{a}_{.q}^{(2,T)}$, $\mathbf{a}_{f.}^{(2,H,\eta*)}$ are the qth column of $\mathbf{T}^*\mathbf{A}_2$ and the fth row of $\mathbf{A}_2^{\eta*}\mathbf{H}$, respectively. Denote

$$\omega_{qj} := \sum_{f=1}^{n} x_{qf}^{(1)} \left(-\eta \sum_{\alpha \in I_{r_3,n}\{j\}} \mathrm{rdet}_j \left((\mathbf{H}^*\mathbf{H})_{j.} \left(\mathbf{a}_{f.}^{(2,H,\eta*)} \right) \right)_{\alpha}^{\alpha} \eta \right) = -\eta \sum_{\alpha \in I_{r_3,n}\{j\}} \mathrm{rdet}_j \left((\mathbf{H}^*\mathbf{H})_{j.} \left(\widehat{\mathbf{x}}_{q.}^{(1,\eta)} \right) \right)_{\alpha}^{\alpha} \eta, \tag{36}$$

where $\widehat{\mathbf{x}}_{q.}^{(1,\eta)}$ is the qth row of $\widehat{\mathbf{X}}_1^{\eta} := \mathbf{X}_1^{\eta}\mathbf{A}_2^{\eta*}\mathbf{H}$. Construct the matrix $\mathbf{\Omega} = (\omega_{qj})$ such that ω_{qj} is determined in (36) and denote $\widehat{\mathbf{\Omega}} := \mathbf{T}^*\mathbf{A}_2\mathbf{\Omega}$. From these denotations and the equation (35), it follows

$$x_{ij}^{(7)} = \frac{\sum_{\beta \in J_{r_4,n}\{i\}} \mathrm{cdet}_i \left((\mathbf{T}^*\mathbf{T})_{.i} (\widehat{\omega}_{.j}) \right)_{\beta}^{\beta}}{\sum_{\beta \in J_{r_4,n}} |\mathbf{T}^*\mathbf{T}|_{\beta}^{\beta} \sum_{\alpha \in I_{r_3,r}} |\mathbf{H}^*\mathbf{H}|_{\alpha}^{\alpha}}, \tag{37}$$

where $\widehat{\omega}_{.j}$ is the jth column of $\widehat{\mathbf{\Omega}}$.

Therefore, we proved the following theorem.

Theorem 3.1. *Suppose that* $\mathbf{A}_1 \in \mathbb{H}_{r_1}^{m\times n}$, $\mathbf{A}_2 \in \mathbb{H}_{r_2}^{k\times n}$, *and* $\mathrm{rank}\mathbf{H} = \mathrm{rank}(\mathbf{A}_2\mathbf{L}_{A_1}) = r_3$, $\mathrm{rank}\mathbf{T} = \mathrm{rank}(\boldsymbol{R}_H\mathbf{A}_2) = r_4$. *The partial solution* (15) *to the system* (2) *by components is*

$$x_{ij} = \sum_{\delta=1}^{4} x_{ij}^{(\delta)} - \sum_{\delta=5}^{7} x_{ij}^{(\delta)}, \quad i,j = 1,\ldots,n,$$

where the summand $x_{ij}^{(1)}$ *has the determinantal representations* (19) *and* (23), $x_{ij}^{(2)}$ – (24) *and* (25), $x_{ij}^{(3)}$ – (26) *and* (27), $x_{ij}^{(4)}$ – (30), $x_{ij}^{(5)}$ – (31), $x_{ij}^{(6)}$ – (34), and $x_{ij}^{(7)}$ – (37).

Remark 3.2. Theorem 3.1 gives the direct method of finding of a general solution to the system (2) that is an analog of Cramer's rule. For this we need in ranks of given matrices and satisfying by them the consistent conditions (13)–(14).Let, now,

$$\mathbf{C}_1 = \mathbf{C}_1^{\eta*}, \quad \mathbf{C}_2 = \mathbf{C}_2^{\eta*}, \tag{38}$$

$$\mathbf{C}_1 = -\mathbf{C}_1^{\eta*}, \quad \mathbf{C}_2 = -\mathbf{C}_2^{\eta*}. \tag{39}$$

The partial η-Hermitian solution $\mathbf{Y}_1$ and η-skew-Hermitian solution $\mathbf{Y}_2$ to the system (2) with the restrictions (38) and (39), respectively, can be expressed as

$$\mathbf{Y}_1 = \tfrac{1}{2}(\mathbf{X} + \mathbf{X}^{\eta*}), \quad \mathbf{Y}_2 = \tfrac{1}{2}(\mathbf{X} - \mathbf{X}^{\eta*}),$$

where $\mathbf{X}$ is an arbitrary solution to the system (2). Due to the expression of the general solution (15) and taking into account (38), we have

$$\begin{aligned}\mathbf{X}^{\eta*} = \left(x_{ij}^{\eta*}\right) = \left(\overline{x}_{ji}{}^{\eta}\right) &= \mathbf{A}_1^{\dagger}\mathbf{C}_1\left(\mathbf{A}_1^{\eta*}\right)^{\dagger} + \mathbf{A}_2^{\dagger}\mathbf{C}_2(\mathbf{H}^{\eta*})^{\dagger} + \mathbf{H}^{\dagger}\mathbf{C}_2(\mathbf{T}^{\eta*})^{\dagger} \\ &+ \mathbf{P}_{A_2}(\mathbf{A}_1^{\eta*})^{\dagger}\mathbf{C}_1\mathbf{A}_1^{\dagger}\mathbf{A}_2(\mathbf{T}^{\eta*})^{\dagger}\mathbf{A}_2^{\eta*}(\mathbf{H}^{\eta*})^{\dagger} - \mathbf{P}_{A_2}(\mathbf{A}_1^{\eta*})^{\dagger}\mathbf{C}_1\mathbf{A}_1^{\dagger}\mathbf{A}_2^{\eta*}(\mathbf{H}^{\eta*})^{\dagger} \\ &- \mathbf{A}_2^{\dagger}\mathbf{C}_2(\mathbf{T}^{\eta*})^{\dagger}\mathbf{A}_2^{\eta*}(\mathbf{H}^{\eta*})^{\dagger} - \mathbf{H}^{\dagger}\mathbf{A}_2\mathbf{A}_1^{\dagger}\mathbf{C}_1(\mathbf{A}_1^{\eta*})^{\dagger}\mathbf{A}_2^{\eta*}(\mathbf{T}^{\eta*})^{\dagger},\end{aligned}$$

But taking into account (39), we obtain

$$\begin{aligned}\mathbf{X}^{\eta*} = -\left(\overline{x}_{ji}{}^{\eta}\right) &= -\mathbf{A}_1^{\dagger}\mathbf{C}_1\left(\mathbf{A}_1^{\eta*}\right)^{\dagger} - \mathbf{A}_2^{\dagger}\mathbf{C}_2(\mathbf{H}^{\eta*})^{\dagger} - \mathbf{H}^{\dagger}\mathbf{C}_2(\mathbf{T}^{\eta*})^{\dagger} \\ &- \mathbf{P}_{A_2}(\mathbf{A}_1^{\eta*})^{\dagger}\mathbf{C}_1\mathbf{A}_1^{\dagger}\mathbf{A}_2(\mathbf{T}^{\eta*})^{\dagger}\mathbf{A}_2^{\eta*}(\mathbf{H}^{\eta*})^{\dagger} + \mathbf{P}_{A_2}(\mathbf{A}_1^{\eta*})^{\dagger}\mathbf{C}_1\mathbf{A}_1^{\dagger}\mathbf{A}_2^{\eta*}(\mathbf{H}^{\eta*})^{\dagger} \\ &+ \mathbf{A}_2^{\dagger}\mathbf{C}_2(\mathbf{T}^{\eta*})^{\dagger}\mathbf{A}_2^{\eta*}(\mathbf{H}^{\eta*})^{\dagger} + \mathbf{H}^{\dagger}\mathbf{A}_2\mathbf{A}_1^{\dagger}\mathbf{C}_1(\mathbf{A}_1^{\eta*})^{\dagger}\mathbf{A}_2^{\eta*}(\mathbf{T}^{\eta*})^{\dagger},\end{aligned}$$

So, the partial η-Hermitian and η-skew-Hermitian solutions to the system (2) with the restrictions (38) and (39), respectively, have the following expression:

$$\begin{aligned}
&\mathbf{Y}_1 = \mathbf{Y}_2 = \\
&\mathbf{A}_1^{\dagger}\mathbf{C}_1\left(\mathbf{A}_1^{\eta*}\right)^{\dagger} + \tfrac{1}{2}\left[\mathbf{H}^{\dagger}\mathbf{C}_2(\mathbf{A}_2^{\eta*})^{\dagger} + \mathbf{A}_2^{\dagger}\mathbf{C}_2(\mathbf{H}^{\eta*})^{\dagger}\right] + \tfrac{1}{2}\left[\mathbf{T}^{\dagger}\mathbf{C}_2(\mathbf{H}^{\eta*})^{\dagger} + \mathbf{H}^{\dagger}\mathbf{C}_2(\mathbf{T}^{\eta*})^{\dagger}\right] \\
&+\tfrac{1}{2}\left[\mathbf{H}^{\dagger}\mathbf{A}_2\mathbf{T}^{\dagger}\mathbf{A}_2\mathbf{A}_1^{\dagger}\mathbf{C}_1(\mathbf{A}_1^{\eta*})^{\dagger}\mathbf{P}_{A_2} + \mathbf{P}_{A_2}(\mathbf{A}_1^{\eta*})^{\dagger}\mathbf{C}_1\mathbf{A}_1^{\dagger}\mathbf{A}_2(\mathbf{T}^{\eta*})^{\dagger}\mathbf{A}_2^{\eta*}(\mathbf{H}^{\eta*})^{\dagger}\right] \\
&-\tfrac{1}{2}\left[\mathbf{H}^{\dagger}\mathbf{A}_2\mathbf{A}_1^{\dagger}\mathbf{C}_1(\mathbf{A}_1^{\eta*})^{\dagger}\mathbf{P}_{A_2} + \mathbf{P}_{A_2}(\mathbf{A}_1^{\eta*})^{\dagger}\mathbf{C}_1\mathbf{A}_1^{\dagger}\mathbf{A}_2^{\eta*}(\mathbf{H}^{\eta*})^{\dagger}\right] \\
&-\tfrac{1}{2}\left[\mathbf{H}^{\dagger}\mathbf{A}_2\mathbf{T}^{\dagger}\mathbf{C}_2(\mathbf{A}_2^{\eta*})^{\dagger} + \mathbf{A}_2^{\dagger}\mathbf{C}_2(\mathbf{T}^{\eta*})^{\dagger}\mathbf{A}_2^{\eta*}(\mathbf{H}^{\eta*})^{\dagger}\right] \\
&-\tfrac{1}{2}\left[\mathbf{T}^{\dagger}\mathbf{A}_2\mathbf{A}_1^{\dagger}\mathbf{C}_1(\mathbf{A}_1^{\eta*})^{\dagger}\mathbf{A}_2^{\eta*}(\mathbf{H}^{\eta*})^{\dagger} + \mathbf{H}^{\dagger}\mathbf{A}_2\mathbf{A}_1^{\dagger}\mathbf{C}_1(\mathbf{A}_1^{\eta*})^{\dagger}\mathbf{A}_2^{\eta*}(\mathbf{T}^{\eta*})^{\dagger}\right].
\end{aligned}$$

The determinantal representations of $\mathbf{Y}_1 = (y_{ij}^{(1)})$ and $\mathbf{Y}_2 = (y_{ij}^{(2)})$ can be obtained by components respectively as

$$y_{ij}^{(1)} = \tfrac{1}{2}\left(x_{ij} + \overline{x}_{ji}{}^{\eta}\right), \quad y_{ij}^{(2)} = \tfrac{1}{2}\left(x_{ij} - \overline{x}_{ji}{}^{\eta}\right),$$

for all $i, j = 1, \ldots, n$, where x_{ij} is determined by Theorem 3.1.

Conclusion

Using row-column noncommutative determinants previously introduced by the author, the determinantal representations (analogs of Cramer's rule) of the general, η-Hermitian and η-skew-Hermitian solutions to the system of quaternion matrix equations $\mathbf{A}_1\mathbf{X}\mathbf{A}_1^{\eta*} = \mathbf{C}_1$, $\mathbf{A}_2\mathbf{X}\mathbf{A}_2^{\eta*} = \mathbf{C}_2$ have been derived. For these purposes, determinantal representations of the Moore–Penrose inverses of a quaternion matrix, its Hermitian adjoint and η-Hermitian adjoint matrices have been explored and used.

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Cite this article as: Kyrchei II 2019. Cramer's rules for the system of quaternion matrix equations with η-Hermicity. 4open, **2**, 24.