4open 2019, **2**, 21

https://doi.org/10.1051/fopen/2019017

Available online at:
www.4open-sciences.org

Mathematical Models
Guest Editors: Fernanda A. Ferreira and Flávio Ferreira

Review Article   Open Access

Euclidean Jordan algebras and some conditions over the spectra of a strongly regular graph

Luís Vieira*

Department of Cívil Engineering of University of Porto, Street Roberto Frias, 0351-4200-465 Porto, Portugal

Received 24 January 2019, Accepted 13 May 2019

Abstract – Let G be a primitive strongly regular graph G such that the regularity is less than half of the order of G and A its matrix of adjacency, and let $\mathcal{A}$ be the real Euclidean Jordan algebra of real symmetric matrices of order n spanned by the identity matrix of order n and the natural powers of A with the usual Jordan product of two symmetric matrices of order n and with the inner product of two matrices being the trace of their Jordan product. Next the spectra of two Hadamard series of $\mathcal{A}$ associated to A^2 is analysed to establish some conditions over the spectra and over the parameters of G.

Keywords: Euclidean Jordan algebras, Strongly regular graphs, Admissibility conditions

Introduction

The Euclidean Jordan algebras have many applications in several areas of mathematics. Some authors applied the theory of Euclidean Jordan algebras to interior-point methods [1–10], others applied this theory to combinatorics [11–14], and to statistics [15–17]. More actually, some authors extended the properties of the real symmetric matrices to the elements of simple real Euclidean Jordan algebras, see [18–24].

A good exposition about Jordan algebras can be founded in the beautiful work of K. McCrimmon, *A taste of Jordan algebras*, see [25], or for a more abstract survey one must cite the work of N. Jacobson, *Structure and Representations of Jordan Algebras*, see [26] and the PhD thesis of Michael Baes, *Spectral Functions and Smoothing Techniques on Jordan Algebras*, see [27].

For a well based understanding of the results of Euclidean Jordan algebras we must cite the works of Faraut and Korányi, *Analysis on Symmetric Cones*, see [28], the PhD thesis *Jordan algebraic approach to symmetric optimization* of Manuel Vieira, see [29], and the PhD thesis *A Gershgorin type theorem, special inequalities and simultaneous stability in Euclidean Jordan algebras* of Melanie Moldovan, see [30].

But for a very readable text on Euclidean Jordan algebras we couldn't avoid of indicating, the chapter written by F. Alizadeh and S. H. Schmieta, *Symmetric Cones, Potential Reduction Methods and Word-By-Word Extensions* of the book Handbook of semi-definite programming, Theory, Algorithms and Applications, see [31], and the chapter, written by F. Alizadeh, *An Introduction to Formally Real Jordan Algebras and Their Applications in Optimization* of the book Handbook on Semidefinite, Conic and Polynomial Optimization, see [32].

In this paper we establish some admissibility conditions in an algebraic asymptotically way over the parameters and over the spectra of a primitive strongly regular graph.

This paper is organized as follows. In the second section we expose the most important concepts and results about Jordan algebras and Euclidean Jordan algebras, without presenting any proof of these results. Nevertheless some bibliography is present on the subject. In the following section we present some concepts about simple graphs and namely strongly regular graphs needed for a clear understanding of this work. In the last section we consider a three dimensional real Euclidean Jordan algebra $\mathcal{A}$ associated to the adjacency matrix of a primitive strongly regular graph and we establish some admissibility conditions over, in an algebraic asymptotic way, the spectra and over the parameters of a strongly regular graph.

*Corresponding author: lvieira@fe.up.pt

Principal results on Euclidean Jordan algebras

Herein, we describe the principal definitions, results and the more relevant theorems of the theory of Euclidean Jordan algebras without presenting the proof of them.

To make this exposition about Euclidean Jordan algebras we have recurred to the the monograph, *Analysis on Symmetric Cones* of Faraut and Kóranyi [28], and to the book *A taste of Jordan algebra* of Kevin McCrimmon [25]. But for general Jordan algebras very readable expositions can be found in the book *Statistical Applications of Jordan algebras* of James D. Malley [17].

Now, we will present only the main results about Euclidean Jordan algebras needed for this paper.

A Jordan algebra $\mathcal{A}$ over a field $\mathbb{K}$ with characteristic $\neq 2$ is a vector space over the field $\mathbb{K}$ with a operation of multiplication $\diamond$ such that for any x and y in $\mathcal{A}$, $x \diamond y = y \diamond x$ and $x \diamond (x^{2\diamond} \diamond y) = x^{2\diamond} \diamond (x \diamond y)$, where $x^{2\diamond} = x \diamond x$. We will suppose throughout this paper that if $\mathcal{A}$ is a Jordan algebra then $\mathcal{A}$ has a unit element that we will denote it by $\mathbf{e}$.

When the field $\mathbb{K}$ is the field of the reals numbers we call the Jordan algebra a real Jordan algebra. Since we are only interested in finite dimensional real Jordan algebras with unit element, we only consider Jordan algebras that are real finite dimensional Jordan algebras and that have an unit element $\mathbf{e}$ and that are equipped with an operation of multiplication that we denote by $\diamond$.

The real vector space of real symmetric matrices, $\mathcal{A} = \mathrm{Sym}\,(n, \mathbb{R})$, of order n, with the operation $x \diamond y = \frac{xy+yx}{2}$ is a real Jordan algebra.

We must note, that we define the powers of an element x in $\mathcal{A}$ in the usual way $x^{0\diamond} = \mathbf{e}$, $x^{1\diamond} = x$, $x^{2\diamond} = x \diamond x$ and $x^{k\diamond} = x \diamond x^{(k-1)\diamond}$ for any natural number k. Hence, we have $x^{2\diamond} = x \diamond x = \frac{xx+xx}{2} = \frac{2x^2}{2} = x^2$ and therefore by induction over $\mathbb{N}$ we conclude that $x^{k\diamond} = x^k$ for any natural number k, where x^k represents the usual power of order k of a squared symmetric matrix. $\mathcal{A}$ is a Jordan algebra since for x and y in $\mathcal{A}$ we have $x \diamond y = \frac{xy+yx}{2} = \frac{yx+xy}{2} = y \diamond x$ and

$$\begin{aligned}
x \diamond (x^{2\diamond} \diamond y) = x \diamond (x^2 \diamond y) = x \diamond \left(\frac{x^2y + yx^2}{2}\right) \\
= \frac{x\left(\frac{x^2y+yx^2}{2}\right) + \left(\frac{x^2y+yx^2}{2}\right)x}{2} = \frac{\frac{x^3y + xyx^2 + x^2yx + yx^3}{2}}{2} \\
= \frac{\frac{x^3y + x^2yx + xyx^2 + yx^3}{2}}{2} = \frac{\frac{x^2(xy+yx) + (xy+yx)x^2}{2}}{2} \\
= \frac{x^2\frac{xy+yx}{2} + \frac{xy+yx}{2}x^2}{2} = \frac{x^2(x \diamond y) + (x \diamond y)x^2}{2} \\
= \frac{x^{2\diamond}(x \diamond y) + (x \diamond y)x^{2\diamond}}{2} = x^{2\diamond} \diamond (x \diamond y).
\end{aligned}$$

Let's consider another example of Euclidean Jordan algebra. Let consider the real vector space $\mathcal{A}_{n+1} = \mathbb{R}^{n+1}$ equipped with the product $\diamond$ such that

$$z \diamond w = \begin{bmatrix} z|w \\ z_1\overline{w} + w_1\overline{x} \end{bmatrix},$$

where $z = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \\ z_{n+1} \end{bmatrix}$, $w = \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \\ w_{n+1} \end{bmatrix}$, $\overline{z} = \begin{bmatrix} z_2 \\ z_3 \\ \vdots \\ z_{n-1} \\ z_n \\ z_{n+1} \end{bmatrix}$ and $\overline{w} = \begin{bmatrix} w_2 \\ w_3 \\ \vdots \\ w_{n-1} \\ w_n \\ w_{n+1} \end{bmatrix}$.

Now we will show that $z \diamond w = w \diamond z$ and that $z \diamond (z^{2\diamond} \diamond w) = z^{2\diamond}(z \diamond w)$. We have

$$z \diamond w = \begin{bmatrix} z|w \\ z_1\overline{w} + w_1\overline{z} \end{bmatrix} = \begin{bmatrix} w|z \\ w_1\overline{z} + z_1\overline{w} \end{bmatrix} = w \diamond z.$$

Herein, we must say that the element $\mathbf{e} = \begin{bmatrix} 1 \\ \overline{0} \end{bmatrix}$ is the unit of the Euclidean Jordan algebra $\mathcal{A}_{n+1}$. Indeed, we have

$$\mathbf{e} \diamond \begin{bmatrix} x_1 \\ \overline{x} \end{bmatrix} = \begin{bmatrix} 1 \\ \overline{0} \end{bmatrix} \diamond \begin{bmatrix} x_1 \\ \overline{x} \end{bmatrix} = \begin{bmatrix} x_1 + \overline{x}|\overline{0} \\ x_1\overline{0} + 1\overline{x} \end{bmatrix} = \begin{bmatrix} x_1 \\ \overline{x} \end{bmatrix}.$$

Now since the operation $\diamond$ is commutative, we showed that

$$\mathbf{e} \diamond \begin{bmatrix} x_1 \\ \overline{x} \end{bmatrix} = \begin{bmatrix} x_1 \\ \overline{x} \end{bmatrix} \diamond \mathbf{e} = \begin{bmatrix} x_1 \\ \overline{x} \end{bmatrix}.$$

Hence, $\mathbf{e}$ is a unit of the Jordan algebra $\mathcal{A}_{n+1}$. Now, we have

$$\begin{aligned} & z \diamond (z^{2\diamond} \diamond w) \\ & = z \diamond ((z \diamond z) \diamond w) \\ & = z \diamond \left(\begin{bmatrix} ||z||^2 \\ 2z_1\overline{z} \end{bmatrix} \diamond \begin{bmatrix} w_1 \\ \overline{w} \end{bmatrix} \right) = \begin{bmatrix} z_1 \\ \overline{z} \end{bmatrix} \diamond \begin{bmatrix} ||z||^2 w_1 + 2z_1\overline{z}|\overline{w} \\ ||z||^2\overline{w} + 2z_1w_1\overline{z} \end{bmatrix} \\ & = \begin{bmatrix} z_1w_1\left(||z||^2 + 2||\overline{z}||^2\right) + (||z||^2 + 2z_1^2)\overline{z}|\overline{w} \\ z_1||z||^2\overline{w} + 2z_1^2w_1\overline{z} + \left(||z||^2w_1 + 2z_1\overline{z}|\overline{w}\right)\overline{z} \end{bmatrix} \\ & = \begin{bmatrix} z_1w_1\left(||z||^2 + 2||\overline{z}||^2\right) + (||z||^2 + 2z_1^2)\overline{z}|\overline{w} \\ z_1||z||^2\overline{w} + \left(2z_1^2w_1 + ||z||^2w_1 + 2z_1\overline{z}|\overline{w}\right)\overline{z} \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} & z^{2\diamond} \diamond (z \diamond w) \\ & = \left(\begin{bmatrix} z_1 \\ \overline{z} \end{bmatrix} \diamond \begin{bmatrix} z_1 \\ \overline{z} \end{bmatrix} \right) \diamond \left(\begin{bmatrix} z_1 \\ \overline{z} \end{bmatrix} \diamond \begin{bmatrix} w_1 \\ \overline{w} \end{bmatrix} \right) = \left(\begin{bmatrix} ||z||^2 \\ 2z_1\overline{z} \end{bmatrix} \right) \diamond \left(\begin{bmatrix} z_1w_1 + \overline{z}|\overline{w} \\ z_1\overline{w} + w_1\overline{z} \end{bmatrix} \right) \\ & = \begin{bmatrix} z_1w_1(||z||^2 + 2||\overline{z}||^2) + (2z_1^2 + ||\overline{z}||^2)\overline{z}|\overline{w} \\ z_1||z||^2\overline{w} + (w_1||z||^2 + 2z_1^2w_1 + 2z_1\overline{z}|\overline{w})\overline{z} \end{bmatrix}. \end{aligned}$$

Therefore, since for any z and w in $\mathcal{A}_{n+1}$ we have $z \diamond w = w \diamond z$ and $z \diamond (z^{2\diamond} \diamond w) = z^2 \diamond (z \diamond w)$. Then we conclude that $\mathcal{A}_{n+1}$ is a Jordan algebra.

Now we define a Jordan subalgebra of the Euclidean Jordan algebra $\mathcal{A} = \mathrm{Sym}\,(n, \mathbb{R})$. Let's consider the set $\{B_1, B_2, \cdots, B_l\}$ of symmetric matrices of $M_n(\mathbb{R})$ such that

(i) $B_1 = I_n$,

(ii) $(B_i)_{jk} \in \{0, 1\}$, for $j, k \in \{1, \cdots, l\}$ and $B_i = B_i^T$ for $i = 1, \cdots, l$,

(iii) $B_1 + B_2 + \cdots + B_l = J_n$,

(iv) $\forall i, j \in \{1, \cdots, l\}, \forall k \in \{1, \cdots, l\} \exists \alpha_{ij}^k \in \mathbb{R} : B_iB_j = \sum_{k=1}^{l} \alpha_{ij}^k B_k$.

Then the real vector space $\mathcal{B}$ spanned by the set $\{B_1, B_2, \cdots, B_l\}$ with the Jordan product $\diamond$ is a Jordan algebra. Firstly, we must say that $\mathcal{B}$ is closed for the Jordan product. Indeed, for C and D in $\mathcal{B}$ we have $C = \alpha_1 B_1 + \cdots + \alpha_l B_l$ and $D = \beta_1 B_1 + \cdots + \beta_l B_l$. Therefore

$$\begin{aligned} C^T = (\alpha_1 B_1 + \cdots + \alpha_l B_l)^T &= \alpha_1 B_1^T + \cdots + \alpha_l B_l^T \\ &= \alpha_1 B_1 + \cdots + \alpha_l B_l = C \end{aligned}$$

and

$$\begin{aligned} D^T = (\beta_1 B_1 + \cdots + \beta_l B_l)^T &= \beta_1 B_1^T + \cdots + \beta_l B_l^T \\ &= \beta_1 B_1 + \cdots + \beta_l B_l = D. \end{aligned}$$

Since $C \diamond D = \frac{CD+DC}{2}$ then

$$(C \diamond D)^T = \frac{(CD + DC)^T}{2} = \frac{(CD)^T + (DC)^T}{2} = \frac{D^T C^T + C^T D^T}{2}$$
$$= \frac{DC + CD}{2} = \frac{CD + DC}{2} = C \diamond D.$$

So $C \diamond D$ is a symmetric matrix of $M_n(\mathbb{R})$, but from property iv) we conclude that $C \diamond D = \sum_{k=1}^{l=1} \gamma_k B_k$, therefore $C \diamond D$ in $\mathcal{B}$. Now, we must say, that for any matrices X and $Y \in \mathcal{B}$ we have $X \diamond Y = Y \diamond X$ and $X \diamond (X^2 \diamond Y) = X^2 \diamond (X \diamond Y)$ since $\mathcal{B}$ is a subalgebra of $\mathcal{A} = \text{Sym}(n, \mathbb{R})$.

Let $\mathcal{A}$ be a n-dimensional Jordan algebra. Then $\mathcal{A}$ is power associative, this is, is an algebra such that for any x in $\mathcal{A}$ the algebra spanned by x and $\mathbf{e}$ is associative. For x in $\mathcal{A}$ we define rank (x) as being the least natural number k such that $\{\mathbf{e}, x^{1\diamond}, \ldots, x^{k\diamond}\}$ is a linearly dependent set and we write rank $(x) = k$. Now since $\dim(\mathcal{A}) = n$ then rank $(x) \leq n$. The rank of A is defined as being the natural number $r = \text{rank}\,(A) = \max\{\text{rank}\,(x) : x \in A\}$. An element x in $\mathcal{A}$ is regular if rank $(x) = r$. Now, we must observe that the set of regular elements of $\mathcal{A}$ is a dense set in $\mathcal{A}$. Let's consider a regular element x of $\mathcal{A}$ and $r = \text{rank}\ (x)$.

Then, there exist real numbers $\alpha_1(x), \alpha_2(x), \ldots, \alpha_{r-1}(x)$ and $\alpha_r(x)$ such that

$$x^{r\diamond} - \alpha_1(x) x^{(r-1)\diamond} + \cdots + (-1)^r \alpha_r(x) \mathbf{e} = 0, \tag{1}$$

where 0 is the zero vector of $\mathcal{A}$. Taking into account (1) we conclude that the polynomial $p(x, -)$ define by the equality (2).

$$p(x, \lambda) = \lambda^r - \alpha_1(x) \lambda^{r-1} + \cdots + (-1)^r \alpha_r(x), \tag{2}$$

is the minimal polynomial of x. When x is a non regular element of $\mathcal{A}$ the minimal polynomial of x has a degree less than r. The polynomial $p(x, -)$ is called the characteristic polynomial of x. Now, we must say that the coefficients $\alpha_i(x)$ are homogeneous polynomials of degree i on the coordinates of x on a fixed basis of $\mathcal{A}$. Since the set of regular elements of $\mathcal{A}$ is a dense set in $\mathcal{A}$ then we extend the definition of characteristic polynomial to non regular elements of $\mathcal{A}$ by continuity. The roots of the characteristic polynomial $p(x, -)$ of x are called the eigenvalues of x. The coefficient $\alpha_1(x)$ of the characteristic polynomial $p(x, -)$ is called the trace of x and we denote it by Trace (x) and we call the coefficient $\alpha_r(x)$ the determinant of x and we denote it by Det (x).

Let $\mathcal{A}$ be a real finite dimensional associative algebra with the bilinear map $(x, y) \mapsto x \diamond y$. We introduce on $\mathcal{A}$ a structure of Jordan algebra by considering a new product $\bullet$ defined by $x \bullet y = (x \diamond y + y \diamond x)/2$ for all x and y in $\mathcal{A}$. The product $\bullet$ is called the Jordan product of x by y. Let $\mathcal{A}$ be a real Jordan algebra and x be a regular element of $\mathcal{A}$. Then we have rank $(x) = r = \text{rank}\,(\mathcal{A})$. We define the linear operator $L_\diamond(x)$ such that $L_\diamond(x)z = x \diamond z, \forall z \in \mathcal{A}$. We define the real vector space $\mathbb{R}[x]$ by $\mathbb{R}[x] = \{z \in \mathcal{A} : \exists \gamma_0, \gamma_1, \cdots, \gamma_{r-1} \in \mathbb{R} : z = \gamma_0 \mathbf{e} + \gamma_1 x^{1\diamond} + \cdots + \gamma_{r-1} x^{(r-1)\diamond}\}$. The restriction of the linear operator $L_\diamond(x)$ to $\mathbb{R}[x]$ we call $L^0_\diamond(x)$. We must note now that trace $(L^0_\diamond(x)) = a_1(x) =$ Trace (x) and det $(L^0_\diamond(x)) = a_r(x) =$ Det (x).

A Jordan algebra is simple if and only if does not contain any nontrivial ideal. An Euclidean Jordan algebra $\mathcal{A}$ is a Jordan algebra with an inner product $\cdot|\cdot$ such that $L_\diamond(x)y|z = y|L_\diamond(x)z$, for all x, y and z in $\mathcal{A}$. Herein, we must say that an Euclidean Jordan algebra is simple if and only if it can't be written as a direct sum of two Euclidean Jordan algebras. But it is already proved that every Euclidean Jordan algebra is a direct orthogonal sum of simple Euclidean Jordan algebras.

The Jordan algebra $\mathcal{A} = \text{Sym}\,(n, \mathbb{R})$ equipped with the Jordan product $x \diamond y = \frac{xy + yx}{2}$ with xy and yx the usual products of matrices of order n, x by y and of y by x and with the inner product $x|y = \text{trace}\,(L_\diamond(x)y)$ for x and y in $\mathcal{A}$ is an Euclidean Jordan algebra. Indeed, we have

$$\begin{aligned}
& L_\diamond(x)y|z = x \diamond y|z \\
& = \text{trace}\,((x \diamond y) \diamond z) \\
& = \text{trace}\left(\left(\tfrac{xy+yx}{2}\right) \diamond z\right) \\
& = \text{trace}\left(\frac{\left(\frac{xy+yx}{2}\right)z + z\left(\frac{xy+yx}{2}\right)}{2}\right) \\
& = \text{trace}\left(\tfrac{(xy)z}{4} + \tfrac{(yx)z}{4} + \tfrac{z(xy)}{4} + \tfrac{z(yx)}{4}\right) \\
& = \text{trace}\left(\tfrac{(xy)z}{4}\right) + \text{trace}\left(\tfrac{(yx)z}{4}\right) + \text{trace}\left(\tfrac{z(xy)}{4}\right) + \text{trace}\left(\tfrac{z(yx)}{4}\right) \\
& = \text{trace}\left(\tfrac{(yx)z}{4}\right) + \text{trace}\left(\tfrac{z(xy)}{4}\right) + \text{trace}\left(\tfrac{(xy)z}{4}\right) + \text{trace}\left(\tfrac{z(yx)}{4}\right) \\
& = \text{trace}\left(\tfrac{y(xz)}{4}\right) + \text{trace}\left(\tfrac{(zx)y}{4}\right) + \text{trace}\left(\tfrac{(xy)z}{4}\right) + \text{trace}\left(\tfrac{(zy)x}{4}\right) \\
& = \text{trace}\left(\tfrac{y(xz)}{4}\right) + \text{trace}\left(\tfrac{y(zx)}{4}\right) + \text{trace}\left(\tfrac{z(xy)}{4}\right) + \text{trace}\left(\tfrac{x(zy)}{4}\right)
\end{aligned}$$

$$\begin{aligned}
&= \text{trace}\left(\tfrac{y(xz)}{4}\right) + \text{trace}\left(\tfrac{y(zx)}{4}\right) + \text{trace}\left(\tfrac{(zx)y}{4}\right) + \text{trace}\left(\tfrac{(xz)y}{4}\right)\\
&= \text{trace}\left(\frac{y\left(\frac{xz+zx}{2}\right)+\left(\frac{xz+zx}{2}\right)y}{2}\right)\\
&= \text{trace}\left(\tfrac{y(x\diamond z)+(x\diamond z)y}{2}\right)\\
&= \text{trace}\,(y\diamond(x\diamond z))\\
&= \text{trace}\,(L_\diamond(y)(x\diamond z))\\
&= y|x\diamond z\\
&= y|L_\diamond(x)z.
\end{aligned}$$

Now, we will show that the $\mathcal{A}_{n+1}=\mathbb{R}^{n+1}$ is an Euclidean Jordan algebra relatively to the inner product $\begin{bmatrix} x_1\\ \overline{x}\end{bmatrix}|\begin{bmatrix} y_1\\ \overline{y}\end{bmatrix}=x_1y_1+\overline{x}|\overline{y}$. We have the following calculations

$$\begin{aligned}
L_\diamond(x)y|z &= (x\diamond y)|z\\
&= \left(\begin{bmatrix} x_1\\ \overline{x}\end{bmatrix}\diamond\begin{bmatrix} y_1\\ \overline{y}\end{bmatrix}\right)|\begin{bmatrix} z_1\\ \overline{z}\end{bmatrix}\\
&= \begin{bmatrix} x_1y_1+\overline{x}|\overline{y}\\ x_1\overline{y}+y_1\overline{x}\end{bmatrix}|\begin{bmatrix} z_1\\ \overline{z}\end{bmatrix}\\
&= x_1y_1z_1+z_1\overline{x}|\overline{y}+x_1\overline{y}|\overline{z}+y_1\overline{x}|\overline{z}
\end{aligned}$$

and

$$\begin{aligned}
y|(L_\diamond(x)z) = y|(x\diamond z) &= \begin{bmatrix} y_1\\ \overline{y}\end{bmatrix}|\left(\begin{bmatrix} x_1\\ \overline{x}\end{bmatrix}\diamond\begin{bmatrix} z_1\\ \overline{z}\end{bmatrix}\right)\\
&\begin{bmatrix} y_1\\ \overline{y}\end{bmatrix}|\begin{bmatrix} x_1z_1+\overline{x}|\overline{z}\\ x_1\overline{z}+z_1\overline{x}\end{bmatrix}\\
&= x_1y_1z_1+y_1\overline{x}|\overline{z}+x_1\overline{y}|\overline{z}+z_1\overline{y}|\overline{x}\\
&= x_1y_1z_1+z_1\overline{y}|\overline{x}+x_1\overline{y}|\overline{z}+y_1\overline{x}|\overline{z}.
\end{aligned}$$

Hence, we conclude that $L_\diamond(x)y|z=y|L_\diamond(x)z$. And therefore $\mathcal{A}_{n+1}$ is an Euclidean Jordan algebra.

Now, we will analyse the rank of the Euclidean Jordan algebra $\mathcal{A}=\text{Sym}\,(n,\mathbb{R})$. Let consider the element x of $\mathcal{A}$ with n distinct eigenvalues λ_1, $\lambda_2,\ldots,\lambda_{n-1}$ and λ_n, and $\mathcal{B}=\{v_1,v_2,\cdots,v_n\}$ an orthonormal basis of $\mathbb{R}^n$ of eigenvectors of x such that $xv_i=\lambda_iv_i$ for i=1,$\cdot\cdot\cdot$, n. Considering the notation $\mathbf{e}=I_n$, then we have:

$$\begin{aligned}
\mathbf{e} &= v_1v_1^T+v_2v_2^T+\cdots+v_nv_n^T\\
x^{1\diamond} &= \lambda_1v_1v_1^T+\lambda_2v_2v_2^T+\cdots+\lambda_nv_nv_n^T\\
x^{2\diamond} &= \lambda_1^2v_1v_1^T+\lambda_2^2v_2^T+\cdots+\lambda_n^2v_nv_n^T\\
\vdots &= \vdots\\
x^{n-1\diamond} &= \lambda^{n-1}v_1v_1^T+\lambda_2^{n-1}v_2v_2^T+\cdots+\lambda_n^{n-1}v_nv_n^T.
\end{aligned}$$

Therefore, the set $\mathcal{X}_{n-1}=\{\mathbf{e},x^{1\diamond},x^{2\diamond},\cdots,x^{n-1\diamond}\}$ is a linearly independent set of $\mathcal{A}$ if and only if the set

$$S_{n-1}=\left\{(1,1,\cdots,1),(\lambda_1,\lambda_2,\cdots,\lambda_n),(\lambda_1^2,\lambda_2^2,\cdots,\lambda_n^2)\cdots,(\lambda_1^{n-1},\lambda_2^{n-1},\cdots,\lambda_n^{n-1})\right\}$$

is a linearly independent set of $\mathbf{R}^n$. But, since

$$\begin{vmatrix} 1 & 1 & \cdots & 1\\ \lambda_1 & \lambda_2 & \cdots & \lambda_n\\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_n^2\\ \vdots & \vdots & \cdots & \vdots\\ \lambda_1^{n-1} & \lambda_2^{n-1} & \cdots & \lambda_n^{n-1}\end{vmatrix} = \prod_{1\le i<j\le n}(\lambda_j-\lambda_i)\neq 0$$

then the set S_{n-1} is a linearly independent set of $\mathbb{R}^n$ and therefore the set

$$\mathcal{X}_{n-1} = \{\mathbf{e}, x^{1\diamond}, x^{2\diamond}, \cdots, x^{n-1\diamond}\}$$

is a linearly independent set of $\mathcal{A}$. The set $\mathcal{X}_n = \{\mathbf{e}, x^{1\diamond}, x^{2\diamond}, \cdots, x^{n\diamond}\}$ is a linear dependent set of $\mathcal{A}$ since the set

$$S_n = \{(1,1,\cdots,1), (\lambda_1, \lambda_2, \cdots, \lambda_n), \cdots, (\lambda_1^{n-1}, \lambda_2^{n-1}, \cdots, \lambda_n^{n-1}), (\lambda_1^n, \lambda_2^n, \cdots, \lambda_n^n)\}$$

is a linearly dependent set of $\mathbb{R}^n$ because the dimension of $\mathbb{R}^n$ is n. Therefore, we conclude that rank $(x) = n$.

Let x be an element of $\mathcal{A}$ with k distinct non null eigenvalues λ_is, and let $v_{i_1}, v_{i_2}, \cdots, v_{i_{l-1}}$ and v_{i_l} be an orthonormal basis of eigenvectors of x of the eigenvector space of x associated λ_i, this is $xv_{i_j} = \lambda_i v_{i_j}, j = 1, \cdots, l_i$. Now, we consider the elements $u_i = \sum_{j=1}^{l_i} v_{i_j}$ and we have $x = \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \cdots + \lambda_k u_k u_k^T$. Therefore, we have

$$\begin{aligned} \mathbf{e} &= u_1 u_1^T + u_2 u_2^T + \cdots + u_k u_k^T \\ x^{1\diamond} &= \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \cdots + \lambda_k u_k u_k^T \\ \vdots &= \vdots \\ x^{k-1\diamond} &= \lambda_1^{k-1} u_1 u_1^T + \lambda_2^{k-1} u_2 u_2^T + \cdots + \lambda_k^{k-1} u_k u_k^T \end{aligned}$$

and, since

$$\begin{vmatrix} 1 & 1 & \cdots & 1 \\ \lambda_1 & \lambda_2 & \cdots & \lambda_k \\ \lambda_1^2 & \lambda_2^2 & \cdots & \lambda_k^2 \\ \vdots & \vdots & \cdots & \vdots \\ \lambda_1^{k-1} & \lambda_2^{k-1} & \cdots & \lambda_k^{k-1} \end{vmatrix} = \prod_{1 \le i < j \le k} (\lambda_j - \lambda_i) \neq 0$$

then the set $\mathcal{X}_{k-1} = \{\mathbf{e}, x^{1\diamond}, x^{2\diamond}, \cdots, x^{k-1\diamond}\}$ is a linearly independent set of $\mathcal{A}$. And, the set $X_k = \{\mathbf{e}, x^{1\diamond}, x^{2\diamond}, \cdots, x^{k-1\diamond}, x^{k\diamond}\}$ is a linearly dependent set of $\mathcal{A}$ since $\dim(\mathbb{R}^k) = k$ and therefore rank $(x) = k$. If x has k distinct eigenvalues where $k-1$ eigenvalues are non null and one is null then one proves one a similar way that rank $(x) = k$.

Therefore we conclude that rank $(\mathcal{A}) = n$ and the regular elements of $\mathcal{A}$ are the elements x of $\mathcal{A}$ with n distinct eigenvalues.

The characteristic polynomial of a regular element of $\mathcal{A}$ is a monic polynomial of minimal degree $n =$ rank (A). Now, let x be an element of $\mathcal{A}$ with n distinct eigenvalues λ_1, $\lambda_2, \cdots$, λ_{n-1} and λ_n then by the Theorem of Cayley-Hamilton we conclude that the polynomial p such that

$$p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$$

is the monic polynomial of minimal degree of x. Therefore since the monic polynomial of minimal degree of element x is unique we conclude that the characteristic polynomial of x, $p(x, -)$ is such that $p(x, \lambda) = p(\lambda)$. This is $p(x, \lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n)$. So, we have $p(x, \lambda) = \lambda^n - (\lambda_1 + \lambda_2 + \cdots + \lambda_n)\lambda^{n-1} + \cdots + (-1)^n \lambda_1 \lambda_2 \cdots \lambda_n$. Therefore, we have Trace $(x) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$ and Det $(x) = \lambda_1 \lambda_2 \cdots \lambda_n$.

Now, we will show that rank $(\mathcal{A}_{n+1}) = 2$ To came to this conclusion, we will firstly show that for $\overline{x} \neq \overline{0}$, rank $\left(\begin{bmatrix} x_1 \\ \overline{x} \end{bmatrix}\right) = 2$ and that for $x_1 \neq 0$, rank $\left(\begin{bmatrix} x_1 \\ \overline{0} \end{bmatrix}\right) = 1$. So, let suppose $\overline{x} \neq \overline{0}$ then, we have

$$\alpha \begin{bmatrix} x_1 \\ \overline{x} \end{bmatrix} + \beta \begin{bmatrix} 1 \\ \overline{0} \end{bmatrix} = \begin{bmatrix} 0 \\ \overline{0} \end{bmatrix} \Leftrightarrow (\alpha = 0) \wedge (\beta = 0).$$

Therefore, the set $\left\{\begin{bmatrix} 1 \\ \overline{0} \end{bmatrix}, \begin{bmatrix} x_1 \\ \overline{x} \end{bmatrix}\right\}$ is a linearly independent set of $\mathcal{A}_{n+1}$. Now, we have $\begin{bmatrix} x_1 \\ \overline{x} \end{bmatrix}^{2\diamond} = \begin{bmatrix} x_1^2 + ||\overline{x}||^2 \\ 2x_1\overline{x} \end{bmatrix}$ and since

$$\begin{bmatrix} x_1^2 + ||\overline{x}||^2 \\ 2x_1\overline{x} \end{bmatrix} = \alpha \begin{bmatrix} x_1 \\ \overline{x} \end{bmatrix} + \beta \begin{bmatrix} 1 \\ \overline{o} \end{bmatrix} \Leftrightarrow (\alpha = 2x_1) \wedge \left(\beta = ||\overline{x}|| - x_1^2\right)$$

we conclude that rank $\left(\begin{bmatrix} x_1 \\ \overline{x} \end{bmatrix}\right) = 2$. If $x_1 \neq 0$, we have $\begin{bmatrix} x_1 \\ \overline{0} \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ \overline{0} \end{bmatrix}$. Then the set $\left\{\begin{bmatrix} 1 \\ \overline{0} \end{bmatrix}, \begin{bmatrix} x_1 \\ \overline{x} \end{bmatrix}\right\}$ is a linearly dependent set of $\mathcal{A}_{n+1}$ then rank $\left(\begin{bmatrix} x_1 \\ \overline{0} \end{bmatrix}\right) = 1$. And, therefore rank $(\mathcal{A}_{n+1}) = 2$. And the regular elements of $\mathcal{A}_{n+1}$ are the elements of $\mathcal{A}_{n+1}$ such that $\overline{x} \neq \overline{0}$.

Since, when $\overline{x} \neq \overline{0}$ we have

$$\begin{bmatrix} x_1 \\ \overline{x} \end{bmatrix}^{2\diamond} - 2x_1 \begin{bmatrix} x_1 \\ \overline{x} \end{bmatrix} + (x_1^2 - ||\overline{x}||^2) \begin{bmatrix} 1 \\ \overline{0} \end{bmatrix} = \begin{bmatrix} 0 \\ \overline{0} \end{bmatrix}.$$

Then, supposing $\overline{x} \neq \overline{0}$ and considering the notation $x = \begin{bmatrix} x_1 \\ \overline{x} \end{bmatrix}$, we conclude that the caractheristic polynomial of x is $p(x, \lambda) = \lambda^2 - 2x_1\lambda + (x_1^2 - ||\overline{x}||^2)$. Therefore, the eigenvalues of x are $\lambda_1(x) = x_1 - ||\overline{x}||$ and $\lambda_2(x) = x_1 + ||\overline{x}||$, Trace $(x) = 2x_1$ and Det $(x) = x_1^2 - ||x||^2$.

Let $\mathcal{A}$ be a real Euclidean Jordan algebra with unit element $\mathbf{e}$. An element f in $\mathcal{A}$ is an idempotent of $\mathcal{A}$ if $f^{2\diamond} = f$. Two idempotents f and g of $\mathcal{A}$ are orthogonal if $f \diamond g = 0$. A set $\{g_1, g_2, \ldots, g_k\}$ of nonzero idempotents is a complete system of orthogonal idempotents of $\mathcal{A}$ if $g_i^{2\diamond} = g_i$, for $i = 1, \ldots, k$, $g_i \diamond g_j = 0$, for $i = j$, and $\sum_{i=1}^{k} g_i = \mathbf{e}$. An element g of $\mathcal{A}$ is a primitive idempotent if it is a non null idempotent of $\mathcal{A}$ and if cannot be written as a sum of two orthogonal nonzero idempotents of $\mathcal{A}$. We say that $\{g_1, g_2, \ldots, g_l\}$ is a Jordan frame of $\mathcal{A}$ if $\{g_1, g_2, \ldots, g_l\}$ is a complete system of orthogonal idempotents such that each idempotent is primitive.

Let consider the matrices E_{jj} of the Euclidean Jordan algebra $\mathcal{A} =$ Sym $(n, \mathbb{R})$ with $j = 1, \cdots, n$ where E_{jj} is the square matrix of order n such that $(E_{jj})_{jj} = 1$ and $(E_{jj})_{ik} = 0$ if $i \neq j$ or $k \neq j$.

Let k be a natural number such that $1 < k < n$. Then $S = \{E_{11} + E_{22} + \cdots + E_{kk}, E_{k+1\,k+1}, \cdots, E_{nn}\}$ is a complete system of orthogonal idempotents of $\mathcal{A}$ and $S' = \{E_{11}, E_{22}, \cdots, E_{nn}\}$ is a Jordan frame of $\mathcal{A}$.

Let consider the Euclidean Jordan algebra $\mathcal{A}_{n+1}$ and x non zero element of $\mathcal{A}_{n+1}$. Then the set $S = \{g_1, g_2\} = \left\{ \frac{1}{2} \begin{bmatrix} 1 \\ -\frac{\overline{x}}{||\overline{x}||} \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 \\ \frac{\overline{x}}{||\overline{x}||} \end{bmatrix} \right\}$ is a Jordan frame of the Euclidean Jordan algebra $\mathcal{A}_{n+1}$. Indeed, we have:

(i) $g_1^{2\diamond} = g_1$ and $g_2^{2\diamond} = g_2$

$$g_1^{2\diamond} = g_1 \diamond g_1 = \frac{1}{2} \begin{bmatrix} 1 \\ -\frac{\overline{x}}{||\overline{x}||} \end{bmatrix} \diamond \frac{1}{2} \begin{bmatrix} 1 \\ -\frac{\overline{x}}{||\overline{x}||} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{4} + \frac{1}{4} \frac{\overline{x}|\overline{x}}{||\overline{x}||^2} \\ -\frac{1}{4} \frac{\overline{x}}{||\overline{x}||} - \frac{1}{4} \frac{\overline{x}}{||\overline{x}||} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ -\frac{\overline{x}}{||\overline{x}||} \end{bmatrix} = g_1$$

and, we have

$$g_2^{2\diamond} = g_2 \diamond g_2 = \frac{1}{2} \begin{bmatrix} 1 \\ \frac{\overline{x}}{||\overline{x}||} \end{bmatrix} \diamond \frac{1}{2} \begin{bmatrix} 1 \\ \frac{\overline{x}}{||\overline{x}||} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{1}{4} + \frac{1}{4} \frac{\overline{x}|\overline{x}}{||\overline{x}||^2} \\ \frac{1}{4} \frac{\overline{x}}{||\overline{x}||} + \frac{1}{4} \frac{\overline{x}}{||\overline{x}||} \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ \frac{\overline{x}}{||\overline{x}||} \end{bmatrix} = g_2$$

(ii)

$$g_1 + g_2 = \frac{1}{2} \begin{bmatrix} 1 \\ -\frac{\overline{x}}{||\overline{x}||} \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ \frac{\overline{x}}{||\overline{x}||} \end{bmatrix} = \begin{bmatrix} 1 \\ \overline{0} \end{bmatrix} = \mathbf{e}$$

(iii)

$$g_1 \diamond g_2 = \begin{bmatrix} 1 \\ -\frac{\overline{x}}{2||\overline{x}||} \end{bmatrix} \diamond \begin{bmatrix} 1 \\ \frac{\overline{x}}{2||\overline{x}||} \end{bmatrix} = \begin{bmatrix} 1 - \frac{||\overline{x}||^2}{||\overline{x}||^2} \\ \frac{\overline{x}}{||\overline{x}||} - \frac{\overline{x}}{||\overline{x}||} \end{bmatrix} \begin{bmatrix} 0 \\ \overline{0} \end{bmatrix}.$$

Therefore we conclude that $\{g_1, g_2\}$ is a Jordan frame of $\mathcal{A}_{n+1}$ since rank $(\mathcal{A}_{n+1}) = 2$.

Theorem 1. ([28], p. 43). *Let $\mathcal{A}$ be a real Euclidean Jordan algebra. Then for x in $\mathcal{A}$ there exist unique real numbers $\beta_1(x), \beta_2(x), \ldots, \beta_k(x)$, all distinct, and a unique complete system of orthogonal idempotents $\{g_1, g_2, \ldots, g_k\}$ such that*

$$x = \beta_1(x)g_1 + \beta_2(x)g_2 + \cdots + \beta_k(x)g_k. \tag{3}$$

The numbers $\beta_j(x)$'s of (3) are the eigenvalues of x and the decomposition (3) is the first spectral decomposition of x.

Theorem 2. ([28], p. 44). *Let $\mathcal{A}$ be a real Euclidean Jordan algebra with rank $(\mathcal{V}) = r$. Then for each* x *in $\mathcal{A}$ there exists a Jordan frame $\{g_1, g_2, \cdots, g_r\}$ and real numbers $\beta_1(x), \cdots, \beta_{r-1}(x)$ and $\beta_r(x)$ such that*

$$x = \beta_1(x)g_1 + \beta_2(x)g_2 + \cdots + \beta_r(x)g_r. \tag{4}$$

Remark 1. The decomposition (4) is called the second spectral decomposition of x. And we have $\text{Det}\,(x) = \prod_{j=1}^{r} \beta_j(x)$, $\text{Trace}\,(x) = \sum_{1}^{r} \beta_j(x)$ and $\alpha_k(x) = \sum_{1 \leq i_1 < \cdots < i_k \leq r} \beta_{i_1}(x) \ldots \beta_{i_k}(x)$.

Example 1. For $\bar{x} \neq 0$, the second spectral decomposition of $x = \begin{bmatrix} x_1 \\ \bar{x} \end{bmatrix}$ relatively to the Jordan frame $S = \{g_1, g_2\} = \left\{ \frac{1}{2} \begin{bmatrix} 1 \\ -\frac{\bar{x}}{||\bar{x}||} \end{bmatrix}, \frac{1}{2} \begin{bmatrix} 1 \\ \frac{\bar{x}}{||\bar{x}||} \end{bmatrix} \right\}$ of $\mathcal{A}_{n+1}$ is

$$x = (x_1 - ||\bar{x}||)g_1 + (x_1 + ||\bar{x}||)g_2.$$

An Euclidean Jordan algebra is called simple if and only if have only trivial ideals.

Any simple Euclidean Jordan algebra is isomorphic to one of the five Euclidean Jordan algebras that we describe below:

(i) The spin Euclidean Jordan algebra $\mathcal{A}_{n+1}$.

(ii) The Euclidean Jordan algebra $\mathcal{A} = \text{Sym}\,(n, \mathbb{R})$ with the Jordan product of matrices and with an inner product of two symmetric matrices as being the trace of their Jordan product.

(iii) The Euclidean Jordan algebra $\mathcal{A} = H_n(\mathbb{C})$ of hermitian matrices of complexes of order n equipped with the Jordan product of two hermitian matrices of complexes and with the scalar product of two Hermitian matrices of complexes as being the real part of the trace of their Jordan product.

(iv) The Euclidean Jordan algebra $\mathcal{A} = H_n(\mathbb{H})$, of hermitian matrices of quaternions of order n equipped with the Jordan product of hermitian matrices of quaternions and with the scalar product of two hermitian matrices of quaternions as being the real part of the trace of their the Jordan product.

(v) The Euclidean Jordan algebra $\mathcal{A} = H_3(\mathbb{O})$ of hermitian matrices of octonions of order n equipped with the Jordan product of two hermitian matrices of octonions and with the inner product of two hermitian matrices of octonions as being the real part of the trace of their Jordan product.

We now describe the Pierce decomposition of an Euclidean Jordan algebra relatively to one of its idempotents. But, first we must say that for any nonzero idempotent g of an Euclidean Jordan algebra $\mathcal{A}$ the eigenvalues of the linear operator $L_\diamond(g)$ are $0, \frac{1}{2}$ and 1 and this fact permits us to say that, considering the eigenspaces $\mathcal{A}(g, 0) = \{x \in \mathcal{A} : L_\diamond(g)(x) = 0x\}, \mathcal{A}\left(g, \frac{1}{2}\right) = \left\{x \in \mathcal{A} : L_\diamond(g)(x) = \frac{1}{2}x\right\}$ and $\mathcal{A}(g, 1) = \{x \in \mathcal{A} : L_\diamond(g)(x) = 1x\}$ of $L(g)$ associated to these eigenvalues we can decompose $\mathcal{A}$ as an orthogonal direct sum $\mathcal{A} = V(g, 0) + V\left(g, \frac{1}{2}\right) + V(g, 1)$. Now, we will describe the Pierce decomposition of the Euclidean Jordan algebra $\mathcal{A} = \text{Sym}\,(n, \mathbb{R})$ relatively to an idempotent of the form

$$C = \begin{bmatrix} I_k & O_{k \times n-k} \\ O^T_{k \times n-k} & O_{n-k \times n-k} \end{bmatrix}.$$

Let $\mathcal{A} = \text{Sym}\,(n, \mathbb{R})$ and $C = \begin{bmatrix} I_k & O_{k \times n-k} \\ \text{O}^T_{k \times n-k} & O_{n-k \times n-k} \end{bmatrix}$. Now, we will show that $\mathcal{A}(C, 1) = \left\{ \begin{bmatrix} X_{k \times k} & O_{k \times n-k} \\ O^T_{k \times n-k} & O_{n-k \times n-k} \end{bmatrix} : \right.$ $\left. X_{k \times k} \in \mathit{Sym}\,(k, \mathbb{R}) \right\}$.

We have

$$\begin{bmatrix} I_k & O_{k\times n-k} \\ O^T_{n-k\times k} & O_{k\times k} \end{bmatrix} \diamond \begin{bmatrix} A_{k\times k} & A_{k\times n-k} \\ A^T_{k\times n-k} & A_{n-k\times n-k} \end{bmatrix} = \begin{bmatrix} A_{k\times k} & A_{k\times n-k} \\ A^T_{k\times n-k} & A_{n-k\times n-k} \end{bmatrix}$$
$$\Updownarrow$$
$$\frac{1}{2}\left(\begin{bmatrix} I_k & O_{k\times n-k} \\ O^T_{k\times n-k} & O_{n-k\times n-k} \end{bmatrix}\begin{bmatrix} A_{k\times k} & A_{k\times n-k} \\ A^T_{k\times n-k} & A_{n-k\times n-k} \end{bmatrix} + \begin{bmatrix} A_{k\times k} & A_{k\times n-k} \\ A^T_{k\times n-k} & A_{n-k\times n-k} \end{bmatrix}\begin{bmatrix} I_k & O_{k\times n-k} \\ O^T_{k\times n-k} & O_{n-k\times n-k} \end{bmatrix}\right) = \begin{bmatrix} A_{k\times k} & A_{k\times n-k} \\ A^T_{k\times n-k} & A_{n-k\times n-k} \end{bmatrix}$$
$$\Updownarrow$$
$$\frac{1}{2}\begin{bmatrix} A_{k\times k} & A_{k\times n-k} \\ O^T_{k\times n-k} & O_{n-k\times n-k} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} A_{k\times k} & O_{k\times n-k} \\ A^T_{k\times n-k} & O_{n-k\times n-k} \end{bmatrix} = \begin{bmatrix} A_{k\times k} & A_{k\times n-k} \\ A^T_{k\times n-k} & A_{n-k\times n-k} \end{bmatrix}$$
$$\Updownarrow$$
$$\begin{bmatrix} A_{k\times k} & \frac{1}{2}A_{k\times n-k} \\ \frac{1}{2}A^T_{k\times n-k} & O_{n-k\times \mathrm{n}-k} \end{bmatrix} = \begin{bmatrix} A_{k\times k} & A_{k\times n-k} \\ A^T_{k\times n-k} & A_{n-k\times n-k} \end{bmatrix}$$
$$\Updownarrow$$
$$(A_{k\times n-k} = O_{k\times n-k}) \wedge (A_{n-k\times n-k} = O_{n-k\times n-k})$$

Therefore, we have

$$\mathcal{A}(C,1) = \left\{ \begin{bmatrix} A_{k\times k} & O_{k\times n-k} \\ O^T_{k\times n-k} & O_{n-k\times n-k} \end{bmatrix} : A_{k\times k} \in M_k(\mathbb{R}) \wedge (A_{k\times k} = A^T_{k\times k}) \right\}$$

Now, let calculate $\mathcal{A}\big(C,\frac{1}{2}\big)$. So, let consider the following equivalences.

$$\begin{bmatrix} I_k & O_{k\times n-k} \\ O^T_{n-k\times k} & O_{k\times k} \end{bmatrix} \diamond \begin{bmatrix} A_{k\times k} & A_{k\times n-k} \\ A^T_{k\times n-k} & A_{n-k\times n-k} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} A_{k\times k} & A_{k\times n-k} \\ A^T_{k\times n-k} & A_{n-k\times n-k} \end{bmatrix}$$
$$\Updownarrow$$
$$\frac{1}{2}\left(\begin{bmatrix} I_k & O_{k\times n-k} \\ O^T_{k\times n-k} & O_{n-k\times n-k} \end{bmatrix}\begin{bmatrix} A_{k\times k} & A_{k\times n-k} \\ A^T_{k\times n-k} & A_{n-k\times n-k} \end{bmatrix} + \begin{bmatrix} A_{k\times k} & A_{k\times n-k} \\ A^T_{k\times n-k} & A_{n-k\times n-k} \end{bmatrix}\begin{bmatrix} I_k & O_{k\times n-k} \\ O^T_{k\times n-k} & O_{n-k\times n-k} \end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix} A_{k\times k} & A_{k\times n-k} \\ A^T_{k\times n-k} & A_{n-k\times n-k} \end{bmatrix}$$
$$\Updownarrow$$
$$\frac{1}{2}\begin{bmatrix} A_{k\times k} & A_{k\times n-k} \\ O^T_{k\times n-k} & O_{n-k\times n-k} \end{bmatrix} + \frac{1}{2}\begin{bmatrix} A_{k\times k} & O_{k\times n-k} \\ A^T_{k\times n-k} & O_{n-k\times n-k} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}A_{k\times k} & \frac{1}{2}A_{k\times n-k} \\ \frac{1}{2}A^T_{k\times n-k} & \frac{1}{2}A_{n-k\times n-k} \end{bmatrix}$$
$$\Updownarrow$$
$$\begin{bmatrix} A_{k\times k} & \frac{1}{2}A_{k\times n-k} \\ \frac{1}{2}A^T_{k\times n-k} & O_{n-k\times n-k} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}A_{k\times k} & \frac{1}{2}A_{k\times n-k} \\ \frac{1}{2}A^T_{k\times n-k} & \frac{1}{2}A_{n-k\times n-k} \end{bmatrix}$$
$$\Updownarrow$$
$$(A_{k\times k} = O_{k\times k}) \wedge (A_{n-k\times n-k} = O_{n-k\times n-k}).$$

Hence, we have $\mathcal{A}(C,\frac{1}{2}) = \left\{ \begin{bmatrix} O_{k\times k} & A_{k\times n-k} \\ A^T_{k\times n-k} & O_{n-k\times n-k} \end{bmatrix} : A_{k\times n-k} \in M_{k\times n-k}(\mathbb{R}) \right\}$. Now, let's calculate $\mathcal{A}(C,0)$. So let consider the following calculations:

$$\begin{bmatrix} I_k & O_{k\times n-k} \\ O^T_{n-k\times k} & O_{k\times k}\end{bmatrix} \diamond \begin{bmatrix} A_{k\times k} & A_{k\times n-k} \\ A^T_{k\times n-k} & A_{n-k\times n-k}\end{bmatrix} = 0\begin{bmatrix} A_{k\times k} & A_{k\times n-k} \\ A^T_{k\times n-k} & A_{n-k\times n-k}\end{bmatrix}$$
$$\Updownarrow$$
$$\tfrac{1}{2}\left(\begin{bmatrix} I_k & O_{k\times n-k} \\ O^T_{k\times n-k} & O_{n-k\times n-k}\end{bmatrix}\begin{bmatrix} A_{k\times k} & A_{k\times n-k} \\ A^T_{k\times n-k} & A_{n-k\times n-k}\end{bmatrix}\right.$$
$$\left.+\begin{bmatrix} A_{k\times k} & A_{k\times n-k} \\ A^T_{k\times n-k} & A_{n-k\times n-k}\end{bmatrix}\begin{bmatrix} I_k & O_{k\times n-k} \\ O^T_{k\times n-k} & O_{n-k\times n-k}\end{bmatrix}\right) = \begin{bmatrix} O_{k\times k} & O_{k\times n-k} \\ O^T_{k\times n-k} & O_{n-k\times n-k}\end{bmatrix}$$
$$\Updownarrow$$
$$\tfrac{1}{2}\begin{bmatrix} A_{k\times k} & A_{k\times n-k} \\ O^T_{k\times n-k} & O_{n-k\times n-k}\end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} A_{k\times k} & O_{k\times n-k} \\ A^T_{k\times n-k} & O_{n-k\times n-k}\end{bmatrix} = \begin{bmatrix} O_{k\times k} & O_{k\times n-k} \\ O^T_{k\times n-k} & O_{n-k\times n-k}\end{bmatrix}$$
$$\Updownarrow$$
$$\begin{bmatrix} A_{k\times k} & \tfrac{1}{2}A_{k\times n-k} \\ \tfrac{1}{2}A^T_{k\times n-k} & O_{n-k\times n-k}\end{bmatrix} = \begin{bmatrix} O_{k\times k} & O_{k\times n-k} \\ O^T_{k\times n-k} & O_{n-k\times n-k}\end{bmatrix}$$
$$\Updownarrow$$
$$(A_{k\times k} = O_{k\times k}) \wedge (A_{k\times n-k} = O_{k\times n-k}).$$

Hence we have $\mathcal{A}(C,0) = \left\{\begin{bmatrix} O_{k\times k} & O_{k\times n-k} \\ O^T_{k\times n-k} & A_{n-k\times n-k}\end{bmatrix} : A_{n-k\times n-k} \in \text{Sym}\,(n-k,\mathbb{R})\right\}$.

For, other hand if we consider a Jordan frame $S = \{g_1, g_2, \cdots, g_r\}$ of an Euclidean Jordan algebra $\mathcal{A}$ and considering the spaces $\mathcal{A}_{ii} = \{x \in \mathcal{A} : L_\diamond(g_i)x = x\}$ and the spaces $\mathcal{A}_{ij} = \{x \in \mathcal{A} : L_\diamond(g_i)x = \frac{1}{2}x \wedge L_\diamond(g_j)x = \frac{1}{2}x\}$ then we obtain the decomposition of $\mathcal{A}$ as an orthogonal direct sum of the vector spaces $\mathcal{A}_{ii}$s and $\mathcal{A}_{ij}$s in the following way: $\mathcal{A} = \sum_{i=1}^{r}\mathcal{A}_{ii} + \sum_{1\leq i<j\leq r}\mathcal{A}_{ij}$.

In the case when the Euclidean symmetric Jordan algebra is $\mathcal{A} = \text{Sym}\,(n,\mathbb{R})$ and we consider the Jordan frame of $\mathcal{A}$, $S = \{E_{11}, E_{22}, \cdots, E_{nn}\}$ we obtain the following spaces $\mathcal{A}_{ii} = \{A \in M_n(\mathbb{R}) : \exists\alpha \in \mathbb{R} : A = \alpha E_{ii}\}$ and the spaces $\mathcal{A}_{ij} = \{A \in M_n(\mathbb{R}) : \exists x_{ij} \in \mathbb{R} : A = x_{ij}(E_{ij} + E_{ji})\}$, where the matrices E_{ii}s are the matrices with 1 in the entry ii and with the others entries zero and the matrix E_{ij} is the matrix with 1 in the entry ij and zero on the others entries. Therefore the Pierce decomposition of $\mathcal{A}$ relatively to the Jordan frame of $\mathcal{A}$ is $\mathcal{A} = \sum_{i=1}^{n}A_{ii} + \sum_{i\leq 1<i<j\leq n}^{n}A_{ij}$. Therefore, we can write, any matrix of the Sym $(n,\mathbb{R})$ in the form $A = \sum_{i=1}^{n}a_{ii}E_{ii} + \sum_{\{1\leq i<j\leq n\}}^{n}a_{ij}(E_{ij} + E_{ji})$.

Let consider the spin Euclidean Jordan algebra $\mathcal{A}_{n+1}$ and let consider the idempotent $c = \frac{1}{2}\begin{bmatrix} 1 \\ \overline{x}\end{bmatrix}$ with $||\overline{x}|| = 1$. We, will obtain the spaces $\mathcal{A}_{n+1}(c,1)$, $\mathcal{A}_{n+1}(c,\frac{1}{2})$ and $\mathcal{A}_{n+1}(c,0)$. We have, the following equivalences

$$\begin{aligned} & c\diamond\begin{bmatrix} a_1 \\ \overline{a}\end{bmatrix} = \begin{bmatrix} a_1 \\ \overline{a}\end{bmatrix} \Leftrightarrow \tfrac{1}{2}\begin{bmatrix} 1 \\ \overline{x}\end{bmatrix}\diamond\begin{bmatrix} a_1 \\ \overline{a}\end{bmatrix} = \begin{bmatrix} a_1 \\ \overline{a}\end{bmatrix} \\ & \Leftrightarrow \begin{bmatrix} \frac{1}{2}a_1 + \frac{1}{2}\overline{x}|\overline{a} \\ \frac{1}{2}\overline{a} + \frac{1}{2}a_1\overline{x}\end{bmatrix} = \begin{bmatrix} a_1 \\ \overline{a}\end{bmatrix} \\ & \Leftrightarrow a_1 = \overline{x}|\overline{a} \wedge \overline{a} = (\overline{x}|\overline{a})\overline{x} \\ & \Leftrightarrow \begin{bmatrix} a_1 \\ \overline{a}\end{bmatrix} = (2\overline{x}|\overline{a})\tfrac{1}{2}\begin{bmatrix} 1 \\ \overline{x}\end{bmatrix} \Leftrightarrow \begin{bmatrix} a_1 \\ \overline{a}\end{bmatrix} = (2\overline{x}|\overline{a})c. \end{aligned}$$

So, we conclude that $\mathcal{A}_{n+1}(c,1) = \{\alpha c, \alpha \in \mathbb{R}\}$. Now, we will calculate $\mathcal{A}_{n+1}(c,0)$.

$$\begin{aligned} & c\diamond\begin{bmatrix} a_1 \\ \overline{a}\end{bmatrix} = 0\begin{bmatrix} a_1 \\ \overline{a}\end{bmatrix} \Leftrightarrow \tfrac{1}{2}\begin{bmatrix} 1 \\ \overline{x}\end{bmatrix}\diamond\begin{bmatrix} a_1 \\ \overline{a}\end{bmatrix} = \begin{bmatrix} 0 \\ \overline{0}\end{bmatrix} \\ & \Leftrightarrow \begin{bmatrix} \frac{1}{2}a_1 + \frac{1}{2}\overline{x}|\overline{a} \\ \frac{1}{2}\overline{a} + \frac{1}{2}a_1\overline{x}\end{bmatrix} = \begin{bmatrix} 0 \\ \overline{0}\end{bmatrix} \Leftrightarrow a_1 = -\overline{x}|\overline{a} \wedge \overline{a} = (\overline{x}|\overline{a})\overline{x} \\ & \Leftrightarrow \begin{bmatrix} a_1 \\ \overline{a}\end{bmatrix} = (-2\overline{x}|\overline{a})\tfrac{1}{2}\begin{bmatrix} 1 \\ -\overline{x}\end{bmatrix}. \end{aligned}$$

Therefore, we obtain, $\mathcal{A}_{n+1}(c,0)=\left\{\alpha\frac{1}{2}\begin{bmatrix}1\\-\overline{x}\end{bmatrix},\alpha\in\mathbb{R}\right\}$. We, can rewrite $\mathcal{A}_{n+1}(c,0)=\left\{\alpha\begin{bmatrix}1 & O_{1\times n}\\ O_{1\times n}^T & -I_n\end{bmatrix}c,\alpha\in\mathbb{R}\right\}$. Now, we will obtain $\mathcal{A}_{n+1}(c,\frac{1}{2})$.
So, we have

$$c\diamond\begin{bmatrix}a_1\\ \overline{a}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}a_1\\ \overline{a}\end{bmatrix}\Leftrightarrow\frac{1}{2}\begin{bmatrix}1\\ \overline{x}\end{bmatrix}\diamond\begin{bmatrix}a_1\\ \overline{a}\end{bmatrix}=\begin{bmatrix}\frac{1}{2}a_1\\ \frac{1}{2}\overline{a}\end{bmatrix}$$

$$\Leftrightarrow\begin{bmatrix}\frac{1}{2}a_1+\frac{1}{2}\overline{x}|\overline{a}\\ \frac{1}{2}\overline{a}+\frac{1}{2}a_1\overline{x}\end{bmatrix}=\begin{bmatrix}\frac{1}{2}a_1\\ \frac{1}{2}\overline{a}\end{bmatrix}\Leftrightarrow\overline{x}|\overline{a}=0\wedge a_1\overline{x}=\overline{0}$$

$$\Leftrightarrow$$

$$(a_1=0)\wedge(\overline{x}|\overline{a}=0).$$

Therefore, we obtain,

$$\mathcal{A}_{n+1}\left(c,\frac{1}{2}\right)=\left\{\alpha\begin{bmatrix}0\\ \overline{a}\end{bmatrix}:\overline{x}|\overline{a}=0,\alpha\in\mathbb{R}\right\}.$$

Now, we will analyse the Pierce decomposition of $\mathcal{A}_{n+1}$ relatively to the Jordan frame

$$\{c_1,c_2\}=\left\{\frac{1}{2}\begin{bmatrix}1\\1\\0_{n-1}\end{bmatrix},\frac{1}{2}\begin{bmatrix}1\\-1\\0_{n-1}\end{bmatrix}\right\}.$$

As in the previous example, we have $(\mathcal{A}_{n+1})_{11}=\mathcal{A}_{n+1}(c_1,1)=\{\alpha c_1:\alpha\in\mathbb{R}\}$, and $(\mathcal{A}_{n+1})_{22}=\mathcal{A}_{n+1}(c_2,1)=\{\alpha c_2:\alpha\in\mathbb{R}\}$. Now, we will calculate the vector space

$$(\mathcal{A}_{n+1})_{12}=\left\{x\in\mathcal{A}_{n+1}:(L_\diamond(c_1)x=\frac{1}{2}x)\wedge(L_\diamond(c_2)x=\frac{1}{2}x)\right\}$$

So, we have the following calculations

$$\frac{1}{2}\begin{bmatrix}1\\1\\ \overline{0}\end{bmatrix}\diamond\begin{bmatrix}x_1\\x_2\\ \overline{x}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}x_1\\x_2\\ \overline{x}\end{bmatrix}\Leftrightarrow\begin{bmatrix}\frac{1}{2}x_1+\frac{1}{2}x_2+\overline{0}|\overline{x}\\ \frac{1}{2}\begin{bmatrix}x_2\\ \overline{x}\end{bmatrix}+\frac{x_1}{2}\begin{bmatrix}1\\ \overline{0}\end{bmatrix}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}x_1\\x_2\\ \overline{x}\end{bmatrix}$$

$$\Leftrightarrow\begin{bmatrix}\frac{1}{2}x_1+\frac{1}{2}x_2\\ \frac{1}{2}x_2+\frac{1}{2}x_1\\ \frac{1}{2}\overline{x}\end{bmatrix}=\begin{bmatrix}\frac{1}{2}x_1\\ \frac{1}{2}x_2\\ \frac{1}{2}\overline{x}\end{bmatrix}$$

$$(x_1=0)\wedge(x_2=0).$$

Therefore $\mathcal{A}_{n+1}(c_1,\frac{1}{2})=\{\begin{bmatrix}0\\0\\ \overline{x}\end{bmatrix}:\overline{x}\in\mathbb{R}^{n-1}\}$. Now, we will calculate $\mathcal{A}_{n+1}(c_2,\frac{1}{2})$.

$$\frac{1}{2}\begin{bmatrix}1\\-1\\ \overline{0}\end{bmatrix}\diamond\begin{bmatrix}x_1\\x_2\\ \overline{x}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}x_1\\x_2\\ \overline{x}\end{bmatrix}\Leftrightarrow\begin{bmatrix}\frac{1}{2}x_1-\frac{1}{2}x_2+\overline{0}|\overline{x}\\ \frac{1}{2}\begin{bmatrix}x_2\\ \overline{x}\end{bmatrix}+x_1\begin{bmatrix}-\frac{1}{2}\\ \overline{0}\end{bmatrix}\end{bmatrix}=\frac{1}{2}\begin{bmatrix}x_1\\x_2\\ \overline{x}\end{bmatrix}$$

$$\Leftrightarrow\begin{bmatrix}\frac{1}{2}x_1-\frac{1}{2}x_2\\ \frac{1}{2}x_2-\frac{1}{2}x_1\\ \overline{x}\end{bmatrix}=\begin{bmatrix}\frac{1}{2}x_1\\ \frac{1}{2}x_2\\ \overline{x}\end{bmatrix}\Leftrightarrow(x_1=0)\wedge(x_2=0).$$

Hence, we have also that $\mathcal{A}_{n+1}(c_2,\frac{1}{2})=\mathcal{A}_{n+1}(c_1,\frac{1}{2})$. Therefore we have

$$(\mathcal{A}_{n+1})_{12}=\left\{\begin{bmatrix}0\\0\\ \overline{x}\end{bmatrix}:\overline{x}\in\mathbb{R}^{n-1}\right\}.$$

Therefore for any element $x=\begin{bmatrix}x_1\\x_2\\ \overline{x}\end{bmatrix}$ of $\mathcal{A}_{n+1}$ we conclude that

$$x = (x_1 + x_2)c_1 + (x_1 - x_2)c_2 + \begin{bmatrix} 0 \\ 0 \\ \bar{x} \end{bmatrix}.$$

A brief introduction to strongly regular graphs

An undirect graph X is a pair of sets $(V(X), E(X))$ with $V(X) = \{v_1, \cdots, v_{n-1}, v_n\}$ and $E(X)$, the set of edges of X, a subset of $V(X) \times V(X)$. For simplicity, we will denote an edge between the vertices a and b by ab. The order of the graph X is the number of vertices of X, $|V(X)|$ and we call the dimension of X, $|E(X)|$ to the number of edges of X. One calls a graph a simple graph if it has no multiple edges (more than one edge between two vertices) and if it has no loops.

Sometimes we make a sketch to represent a graph X like the one presented on the Figure 1.

An edge is incident on a vertice v of a graph X if v is one of its extreme points. Two vertices of a graph X are adjacent if they are connected by an edge. The adjacency matrix of a simple graph X of order n is a square matrix of order n, A such that $A = [a_{ij}]$ where $a_{ij} = 1$ if $v_i v_j \in E(X)$ and 0 otherwise. The adjacency matrix of a simple graph is a symmetric matrix and we must observe that the diagonal entries of this matrix are null. The number of edges incident to a vertice v of a simple graph is called the degree of v. And, we call a simple graph a regular graph if each of its vertices have the same degree and we say that a graph G is a k-regular graph if each of its vertices have degree k.

The complement graph of a graph X denoted by $\overline{X}$ is a graph with the same set of vertices of X and such that two distinct vertices are adjacent vertices of $\overline{X}$ if and only if they are non adjacent vertices of X.

Along this paper we consider only non-empty, simple and non complete graphs.

Strongly regular graphs were firstly introduced by R. C. Bose in the paper [33].

A graph X is called a $(n, k; \lambda, \mu)$-strongly regular graph if is k-regular and any pair of adjacent vertices have λ common neighbors and any pair of non-adjacent vertices have μ common adjacent vertices.

The adjacency matrix A of a $(n, k; \lambda, \mu)$-strongly regular X satisfies the equation $A^2 = kI_n + \lambda A + \mu(J_n - A - I_n)$, where J_n is the all ones real matrix of order n.

The eigenvalues θ, τ, k and the multiplicities m_θ and m_τ of θ and τ respectively of a $(n, k; \lambda, \mu)$-strongly regular graph X, see, for instance [34, 35], are defined by the equalities (5):

$$\begin{aligned} \theta &= (\lambda - \mu + \sqrt{(\lambda - \mu)^2 + 4(k - \mu)})/2, \\ \tau &= (\lambda - \mu - \sqrt{(\lambda - \mu)^2 + 4(k - \mu)})/2, \\ m_\theta &= \frac{|\tau| n + \tau - k}{n(\theta - \tau)}, \\ m_\tau &= \frac{\theta n + k - \theta}{n(\theta - \tau)}. \end{aligned} \tag{5}$$

Therefore necessary conditions for the parameters of a $(n, k; \lambda, \mu)$-strongly regular graph are that $\frac{|\tau| n + \tau - k}{n(\theta - \tau)}$ and $\frac{\theta n + k - \theta}{n(\theta - \tau)}$ must be natural numbers, they are known as the integrability conditions of a strongly regular graph, see [34].

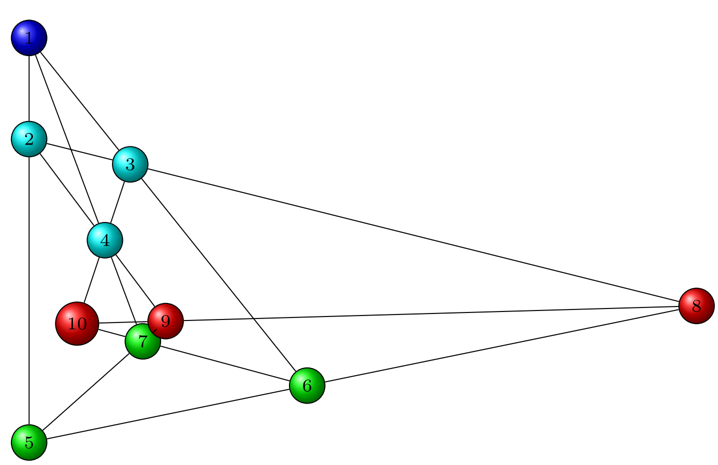

Figure 1. A simple graph X.

We note that a graph X is a $(n,\ k;\ \lambda,\ \mu)$-strongly regular graph if and only if its complement graph $\overline{X}$ is a $(n,\ n-k-1;\ n-2-2k+\mu,\ n-2k+\lambda)$. Now we will present some admissibility conditions on the parameters of a $(n,\ k;\ \lambda,\ \mu)$-strongly regular graph. The parameters of a $(n,\ k;\ \lambda,\ \mu)$-strongly regular graph X verifies the admissibility condition (6).

$$k(k-1-\lambda) = \mu(n-k-1). \tag{6}$$

The inequalities (7) are known as the Krein conditions of X, see [36].

$$\begin{aligned}(k+\theta)(\tau+1)^2 &\geq (\theta+1)(k+\theta+2\theta\tau),\\ (k+\tau)(\theta+1)^2 &\geq (\tau+1)(k+\tau+2\theta\tau).\end{aligned} \tag{7}$$

Given a graph X, we call a path in X between two vertices v_1 and v_{k+1} to a non null sequence of distinct vertices, exceptionally the first vertice and the last vertice can be equal, and distinct edges, $W = v_1 e_1 v_2 e_2 v_3 \cdots v_k e_k v_{k+1}$ whose terms are vertices and edges alternated and such that for $1 \leq i \leq k$ the vertices v_i and v_{i+1} define the edge e_i. The path is a closed path or a cycle if and only if the only repeated vertices are the initial vertice and the final vertice.

A graph Y' is a subgraph of a graph Y and we write $Y' \sqsubseteq Y$ if $V(Y') \sqsubseteq V(Y)$ and $E(Y') \sqsubseteq E(Y)$. If $Y' \sqsubseteq Y$ and $Y' \neq Y$, we say that Y' is a proper subgraph of Y. We must observe that for any non empty subset V' of $V(Y)$ we can construct a subgraph of Y whose set of vertices is V' and whose set of edges is formed by the edges of $E(Y)$ whose extreme points are vertices in V' which we call the induced subgraph of Y and which we denote by $Y(V')$. Two vertices v_1 and v_2 of a graph X are connected if there is a path between v_1 and v_2 in X. This relation between vertices is a relation of equivalence in the set of vertices of the graph X, $V(X)$, whereby there exists a partition of $V(X)$ in non empty subsets $V_1,\ V_2, \cdots,\ V_\mathrm{l}$ of $V(X)$ such that two vertices are connected if and only if they belong to the same set V_i for a given $i \in \{1, 2, \cdots, l\}$. The subgraphs $X(V_1), X(V_2), \cdots, X(V_l)$ are called the connected components of X. If X has only one component then we say that the graph X is connected otherwise the graph X is a disconnected graph. A $(n,\ k;\ \lambda,\ \mu)$-strongly regular graph X is primitive if and only X and $\overline{X}$ are connected graphs. Otherwise is called an imprimitive strongly regular graph. To characterize the connected graphs we will present the definition of reducible and of irreducible matrix. Let n be a natural number greater or equal 2 and A a matrix in $M_n(\mathbb{R})$. We say that the matrix A is a reducible matrix if there exists a permutation matrix P such that

$$P^T A P = \begin{bmatrix} C_{k\times k} & O_{k\times n-k} \\ D_{n-k\times k} & E_{n-k\times n-k} \end{bmatrix} \tag{8}$$

where k is such that $1 \leq k \leq n-1$, if doesn't exist such matrix P we say that the matrix A is irreducible. If A is a reducible symmetric of order n then $D_{n-k\times k} = \mathrm{O}_{n-k\times k}$. From the Theorem of Frobenius we know that if A is a real square irreducible matrix with non negative entries then A has an eigenvector u such that $Au = ru$ with all entries positive and such that $|\lambda| \leq r$ for any eigenvalue of A and r is a simple positive eigenvalue of A. Now since a graph is connected if and only if its matrix of adjacency is irreducible then, if a graph X is a connected strongly regular graph then the greater eigenvalue of its adjacency matrix A is a simple eigenvalue of A with an eigenvector with all components positive. Hence the regularity of a connected strongly regular graph X is a simple eigenvalue of the adjacency matrix of X.

Finally, since from now on, we only consider primitive strongly regular graphs, we note that a $(n,\ k;\ \lambda,\ \mu)$-strongly regular graph is imprimitive if and only if $\mu = 0$ or $\mu = k$. From now on, we only consider $(n,\ k;\ \lambda,\ \mu)$- strongly regular graphs with $k-1 \geq \mu > 0$. The multiplicities of the eigenvalues θ and τ of a primitive strongly regular graph X of order n satisfy the conditions (10) known has the absolute bounds

$$\frac{m_\theta(m_\theta+3)}{2} \geq n \tag{9}$$

$$\frac{m_\tau(m_\tau+3)}{2} \geq n. \tag{10}$$

and they also satisfy the equalities $1 + m_\theta + m_\tau = n$ and $k + m_\theta\theta + m_\tau\tau = 0$.

Some relations on the parameters of a strongly regular graph

Let m be a natural number. We denote the set of real matrices of order m by $M_m(\mathbb{R})$ and the set of symmetric of $M_m(\mathbb{R})$ by Sym $(m, \mathbb{R})$. For any two matrices $H = [h_{ij}]$ and $L = [l_{ij}]$ of $M_m(\mathbb{R})$. we define the Hadamard product of H and L as being the matrix $H \circ L = [h_{ij} l_{ij}]$ and the Kronecker product of matrices H and L as being the matrix $H \otimes L = [h_{ij} L]$. For any matrix P of $M_m(\mathbb{R})$ and for any nonnegative integer number j we define the Schur (Hadamard) power of order j of P, as being the matrix $P^{j\circ}$ in the following way: $P^{0\circ} = J_n, P^{1\circ} = P$ for any natural number $j \geq 2$ we define $P^{j+1\circ} = P \circ P^{j\circ}$.

In this section we will establish some inequalities over the parameters and over the spectra of a primitive strongly regular graph.

Let's consider a primitive $(n, k; \lambda, \mu)$-strongly regular graph G such that $\frac{n}{2} > k > \mu > 0$ and with $\mu < \lambda$, and let A be its adjacency matrix with the distinct eigenvalues τ, θ and k. Now, we consider the Euclidean Jordan algebra $\mathcal{A} = \mathrm{Sym}(n, \mathbb{R})$ with the Jordan product $x \diamond y = \frac{xy+yx}{2}$ and with the inner product $x|y = \mathrm{trace}(x \diamond y)$, where xy and yx are the usual products of x by y and the usual product of y by x. Now we consider the Euclidean Jordan subalgebra $\mathcal{A}$ of $\mathrm{Sym}(n, \mathbb{R})$ spanned by I_n and the natural powers of A. We have that rank $(A) = 3$ since has three distinct eigenvalues and is a three dimensional real Euclidean Jordan algebra. Let $\mathcal{B} = \{E_1, E_2, E_3\}$ be the unique Jordan frame of $\mathcal{A}$ associated to A, where

$$
\begin{aligned}
E_1 &= \frac{1}{n}I_n + \frac{1}{n}A + \frac{1}{n}(J_n - A - I_n),\\
E_2 &= \frac{|\tau|n + \tau - k}{n(\theta-\tau)}I_n + \frac{n+\tau-k}{n(\theta-\tau)}A + \frac{\tau-k}{n(\theta-\tau)}(J_n - A - I_n),\\
E_3 &= \frac{\theta n + k - \theta}{n(\theta-\tau)}I_n + \frac{-n+k-\theta}{n(\theta-\tau)}A + \frac{k-\theta}{n(\theta-\tau)}(J_n - A - I_n).
\end{aligned}
$$

Let's consider the real positive number x such that $x \leq 1$, and let's consider the binomial Hadamard series $S_z = \sum_{l=0}^{+\infty}(-1)^l\binom{-z}{l}\left(\frac{(A^2-\theta^2 I_n)^{2\circ}}{k^4}\right)^{l\circ}$. The second spectral decomposition of S_z relatively to the Jordan frame $\mathcal{B}$ is $S_z = \sum_{i=1}^{3} q_{iz}E_i$. Now, we show that the eigenvalues q_{iz} of S_z are positive.

Since $(-1)^l\binom{-z}{l} = (-1)^l\frac{(-z)(-z-1)(-z-2)\cdots(-z-l+1)}{l!}$ then

$$(-1)^l\binom{-z}{l} = (-1)^{2l}\frac{(z)(z+1)(z+2)\cdots(z+l-1)}{l!} \geq 0.$$

Now, we have $S_{nz} = \sum_{l=0}^{n}(-1)^l\binom{-z}{l}\left(\frac{(A^2-\theta^2 I_n)^{2\circ}}{k^4}\right)^{l\circ}$. Since $A^2 = kI_n + \lambda A + \mu(J_n - A - I_n)$ then we conclude that

$$
\begin{aligned}
\frac{(A^2-\theta^2 I_n)^{2\circ}}{k^4} &= \frac{(kI_n + \lambda A + \mu(J_n - A - I_n) - \theta^2 I_n)^{2\circ}}{k^4}\\
&= \frac{((k-\theta^2)I_n + \lambda A + \mu(J_n - A - I_n))^{2\circ}}{k^4}\\
&= \frac{(k-\theta^2)^2 I_n + \lambda^2 A + \mu^2(J_n - A - I_n)}{k^4}\\
&= \frac{(k-\theta^2)^2}{k^4}I_n + \frac{\lambda^2}{k^4}A + \frac{\mu^2}{k^4}(J_{\mathrm{n}} - A - I_n).
\end{aligned}
$$

Let's consider the second spectral decomposition $S_{nz} = q_{n1z}E_1 + q_{n2z}E_2 + q_{n3z}E_3$. Since $\lambda > \mu$ then we have that $|\tau| < \theta$ and therefore $|\tau|^2 < \theta^2$ and since $\theta^2 \leq k^2$ then the eigenvalues of $\frac{A^2-\theta^2 I_n}{k^4}$ are positive. Since for any two matrices C and D of $M_n(\mathbb{R})$ we have $\lambda_{\min}(C)\lambda_{\min}(D) \leq \lambda_{\min}(C \circ D)$ and since B is a Jordan frame of $\mathcal{A}$ that is a basis of $\mathcal{A}$ and $\mathcal{A}$ is closed for the Schur product of matrices we deduce that the eigenvalues of $\left(\frac{(A^2-\theta^2 I_n)^{2\circ}}{k^4}\right)^{l\circ}$ are positive. So we conclude that the eigenvalues $q_{\mathrm{n}iz}$ for $i = 1, \cdots, 3$ of $S_{\mathrm{n}z}$ are all positive.

Since $q_{1z} = \lim_{n\to+\infty} q_{n1z}$, $q_{2z} = \lim_{n\to+\infty} q_{n2z}$, $q_{3z} = \lim_{n\to+\infty} q_{n3z}$ then we have $q_{1z} \geq 0, q_{2z} \geq 0$ and $q_{3z} \geq 0$. We must say that $S_zE_1 = q_{1z}E_1, S_zE_2 = q_{2z}E_2$ and $S_zE_3 = q_{3z}E_3$, hence we have: $q_{1z} = \frac{1}{(\frac{k^4-(k-\theta^2)^2}{k^4})^z} + \frac{1}{(\frac{k^4-\lambda^2}{k^4})^z}k + \frac{1}{(\frac{k^4-\mu^2}{k^4})^z}(n-k-1)$, $q_{2z} = \frac{1}{(\frac{k^4-(k-\theta^2)^2}{k^4})^z} + \frac{1}{(\frac{k^4-\lambda^2}{k^4})^z}\theta + \frac{1}{(\frac{k^4-\mu^2}{k^4})^z}(-\theta-1)$ and $q_{3z} = \frac{1}{(\frac{k^4-(k-\theta^2)^2}{k^4})^z} + \frac{1}{(\frac{k^4-\lambda^2}{k^4})^z}\tau + \frac{1}{(\frac{k^4-\mu^2}{k^4})^z}(-\tau-1)$.

Let's consider the element $S_{3z} = E_3 \circ S_z$ of $\mathcal{A}$. Since the eigenvalues of E_3 and of S_z are positive and since $\lambda_{\min}(E_3)\lambda_{\min}(S_z) \leq \lambda_{\min}(E_3 \circ S_x)$ then the eigenvalues of $E_3 \circ S_z$ are positive. Now since $k < \frac{n}{2}$ and $\lambda > \mu$ and by an asymptotical analysis of the spectrum of $E_3 \circ S_z$ we will deduce the inequalities (16) and (21) of the Theorems 3 and 4 respectively are verified.

Now, we consider the second spectral decomposition $E_3 \circ S_z = q_{3z}^1 E_1 + q_{3z}^2 E_2 + q_{3z}^3 E_3$. Then, we have

$$q_{3z}^1 = \frac{\theta n + k - \theta}{n(\theta-\tau)}\frac{1}{\left(\frac{k^4-(k-\theta^2)^2}{k^4}\right)^z} + \frac{-n+k-\theta}{n(\theta-\tau)}\frac{1}{\left(\frac{k^4-\lambda^2}{k^4}\right)^z}k + \frac{k-\theta}{n(\theta-\tau)}\frac{1}{\left(\frac{k^4-\mu^2}{k^4}\right)^z}(n-k-1), \tag{11}$$

$$q_{3z}^2 = \frac{\theta n + k - \theta}{n(\theta - \tau)} \frac{1}{\left(\frac{k^4-(k-\theta^2)^2}{k^4}\right)^z} + \frac{-n+k-\theta}{n(\theta-\tau)} \frac{1}{\left(\frac{k^4-\lambda^2}{k^4}\right)^z} \tau + \frac{k-\theta}{n(\theta-\tau)} \frac{1}{\left(\frac{k^4-\mu^2}{k^4}\right)^z} (-\tau - 1), \tag{12}$$

$$q_{3z}^3 = \frac{\theta n + k - \theta}{n(\theta - \tau)} \frac{1}{\left(\frac{k^4-(k-\theta^2)^2}{k^4}\right)^z} + \frac{-n+k-\theta}{n(\theta-\tau)} \frac{1}{\left(\frac{k^4-\lambda^2}{k^4}\right)^z} \theta + \frac{k-\theta}{n(\theta-\tau)} \frac{1}{\left(\frac{k^4-\mu^2}{k^4}\right)^z} (-\theta - 1). \tag{13}$$

Now, since $\frac{\theta n+k-\theta}{n(\theta-\tau)} + \frac{-n+k-\theta}{n(\theta-\tau)} k + \frac{k-\theta}{n(\theta-\tau)} (n-k-1) = 0$ then $\frac{k-\theta}{n(\theta-\tau)} (n-k-1) = -\frac{\theta n+k-\theta}{n(\theta-\tau)} - \frac{-n+k-\theta}{n(\theta-\tau)} k$ and consequently the parameter q_{3z}^1 takes the form:

$$q_{3z}^1 = \frac{\theta n + k - \theta}{n(\theta-\tau)} \left(\frac{\left(\frac{k^4-\mu^2}{k^4}\right)^z - \left(\frac{k^4-(k-\theta^2)^2}{k^4}\right)^z}{\left(1 - \frac{(k-\theta^2)^2}{k^4}\right)^z \left(1 - \frac{\mu^2}{k^4}\right)^z} \right) + \frac{-n+k-\theta}{n(\theta-\tau)} \left(\frac{\left(\frac{k^4-\mu^2}{k^4}\right)^z - \left(\frac{k^4-\lambda^2}{k^4}\right)^z}{\left(1 - \frac{\lambda^2}{k^4}\right)^z \left(1 - \frac{\mu^2}{k^4}\right)^z} - \right) k, \tag{14}$$

and q_{3z}^3 takes the form:

$$q_{3z}^3 = \frac{\theta n + k - \theta}{n(\theta-\tau)} \left(\frac{\left(\frac{k^4-\mu^2}{k^4}\right)^z - \left(\frac{k^4-(k-\theta^2)^2}{k^4}\right)^x}{\left(\frac{k^4-(k-\theta^2)^2}{k^4}\right)^z \left(\frac{k^4-\mu^2}{k^4}\right)^z} \right) + \frac{-n+k-\theta}{n(\theta-\tau)} \left(\frac{\left(\frac{k^4-\mu^2}{k^4}\right)^z - \left(\frac{k^4-\lambda^2}{k^4}\right)^z}{\left(\frac{k^4-\lambda^2}{k^4}\right)^z \left(\frac{k^4-\mu^2}{k^4}\right)^z} \right) \theta. \tag{15}$$

Using the fact that $k < \frac{n}{2}$ then we conclude that $\frac{n-k+\theta}{\theta n+k-\theta} > \frac{1}{2\theta+1}$.

Theorem 3. *Let n, k, μ and λ be natural numbers with n − 1 > k > μ and X be a (n, k; λ, μ)-primitive strongly regular graph with distinct eigenvalues k, θ and τ. If* $k < \frac{n}{2}$ *and* $\lambda > \mu$ *then*

$$\left(\frac{k^4-\mu^2}{k^4-(k-\theta^2)^2}\right)^{2\theta+1} > \left(\frac{k^4-\mu^2}{k^4-\lambda^2}\right)^k \tag{16}$$

Proof. Since $q_{3z}^1 \geq 0$ and recurring to the equality (14) we deduce the equality (17).

$$\frac{\theta n + k - \theta}{n(\theta-\tau)} \cdot \left(\frac{\left(\frac{k^4-\mu^2}{k^4}\right)^z - \left(\frac{k^4-(k-\theta^2)^2}{k^4}\right)^z}{\left(\frac{k^4-(k-\theta^2)^2}{k^4}\right)^z \left(\frac{k^4-\mu^2}{k^4}\right)^z} \right) + \frac{-n+k-\theta}{n(\theta-\tau)} k \left(\frac{\left(\frac{k^4-\mu^2}{k^4}\right)^z - \left(\frac{k^4-\lambda^2}{k^4}\right)^z}{\left(\frac{k^4-\lambda^2}{k^4}\right)^z \left(\frac{k^4-\mu^2}{k^4}\right)^z} \right) \geq 0. \tag{17}$$

Making, some algebraic manipulation of equality (17) we obtain the inequality (18).

$$\frac{\left(\frac{k^4-\lambda^2}{k^4}\right)^z}{\left(\frac{k^4-(k-\theta^2)^2}{k^4}\right)^z} \geq \frac{n-k+\theta}{\theta n+k-\theta} k \left(\frac{\left(\frac{k^4-\mu^2}{k^4}\right)^z - \left(\frac{k^4-\lambda^2}{k^4}\right)^z}{\left(\frac{k^4-\mu^2}{k^4}\right)^z - \left(\frac{k^4-(k-\theta^2)^2}{k^4}\right)^z} \right). \tag{18}$$

Calculating the limits of the expressions of both hand sides of (18) when x approaches zero we obtain the equality (19).

$$1 \geq \frac{n-k+\theta}{\theta n+k-\theta} k \left(\frac{\ln\left(\frac{k^4-\mu^2}{k^4}\right) - \ln\left(\frac{k^4-\lambda^2}{k^4}\right)}{\ln\left(\frac{k^4-\mu^2}{k^4}\right) - \ln\left(\frac{k^4-(k-\theta^2)^2}{k^4}\right)} \right). \tag{19}$$

Now since when $k < \frac{n}{2}$ we have $\frac{n-k+\theta}{\theta n+k-\theta} > \frac{1}{2\theta+1}$ and after some algebraic manipulation of the inequality (19) we obtain the inequality (20).

$$1 > \frac{k}{2\theta+1} \left(\frac{\ln\left(\frac{k^4-\mu^2}{k^4-\lambda^2}\right)}{\ln\left(\frac{k^4-\mu^2}{k^4-(k-\theta^2)^2}\right)} \right). \tag{20}$$

And, finally we conclude that $\left(\frac{k^4-\mu^2}{k^4-(k-\theta^2)^2}\right)^{2\theta+1} > \left(\frac{k^4-\mu^2}{k^4-\lambda^2}\right)^k$.

Theorem 4. *Let n, k, μ and λ be natural numbers and let X be a (n, k; λ, μ)-primitive strongly regular graph with* $n-1 > k > \mu, \frac{n}{2} > k, \mu < \lambda$ *and with the distinct eigenvalues θ, τ and k. Then*

$$\left(\frac{k^4-\mu^2}{k^4-(k-\theta^2)^2}\right)^{2\theta+1} > \left(\frac{k^4-\mu^2}{k^4-\lambda^2}\right)^{\theta}. \tag{21}$$

Proof. Now since $q_{3z}^3 \geq 0$ and recurring to the equality (15) we obtain that

$$\frac{\theta n+k-\theta}{n(\theta-\tau)}\cdot\left(\frac{\left(\frac{k^4-\mu^2}{k^4}\right)^z-\left(\frac{k^4-(k-\theta^2)^2}{k^4}\right)^z}{\left(\frac{k^4-(k-\theta^2)^2}{k^4}\right)^z\left(\frac{k^4-\mu^2}{k^4}\right)^z}\right)+\frac{-n+k-\theta}{n(\theta-\tau)}\left(\frac{\left(\frac{k^4-\mu^2}{k^4}\right)^z-\left(\frac{k^4-\lambda^2}{k^4}\right)^z}{\left(\frac{k^4-\lambda^2}{k^4}\right)^z\left(\frac{k^4-\mu^2}{k^4}\right)^z}\right)\theta \geq 0. \tag{22}$$

From (22) we deduce the inequality (23).

$$\frac{\left(\frac{k^4-\lambda^2}{k^4}\right)^z}{\left(\frac{k^4-(k-\theta^2)^2}{k^4}\right)^z} \geq \frac{n-k+\theta}{\theta n+k-\theta}\left(\frac{\left(\frac{k^4-\mu^2}{k^4}\right)^z-\left(\frac{k^4-\lambda^2}{k^4}\right)^z}{\left(\frac{k^4-\mu^2}{k^4}\right)^z-\left(\frac{k^4-(k-\theta^2)^2}{k^4}\right)^z}\right)\theta. \tag{23}$$

Calculating the limits of the expressions of both hand sides of (23) when x approaches zero we obtain the equality (24).

$$1 \geq \frac{n-k+\theta}{\theta n+k-\theta}\theta\left(\frac{\ln\left(\frac{k^4-\mu^2}{k^4}\right)-\ln\left(1-\frac{k^4-\lambda^2}{k^4}\right)}{\ln\left(\frac{k^4-\mu^2}{k^4}\right)-\ln\left(\frac{k^4-(k-\theta^2)^2}{k^4}\right)}\right). \tag{24}$$

Now, since when $k<\frac{n}{2}$ we have $\frac{n-k+\theta}{\theta n+k-\theta} > \frac{1}{2\theta+1}$ and after some algebraic manipulation of the inequality (24) we obtain the inequality (25).

$$1 > \frac{\theta}{2\theta+1}\left(\frac{\ln\left(\frac{k^4-\mu^2}{k^4-\lambda^2}\right)}{\ln\left(\frac{k^4-\mu^2}{k^4-(k-\theta^2)^2}\right)}\right). \tag{25}$$

And, finally we conclude that $\left(\frac{k^4-\mu^2}{k^4-(k-\theta^2)^2}\right)^{2\theta+1} > \left(\frac{k^4-\mu^2}{k^4-\lambda^2}\right)^{\theta}$.

Preliminares on quaternions and octonions

This section is a brief introduction on quaternions and octonions. Good readable texts on this algebraic structures can be found on the works [37, 38]. Now, we consider the real linear space A of quaternions spanned by the basis $B=\{1,i,j,k\}$, where the elements of B verify the following rules of multiplication $i^2=j^2=k^2=-1$ and

1) $ij=-ji=k$;
2) $jk=-kj=i$;
3) $ki=-ik=j$.

So, we can write $\mathcal{A}=\{\alpha_0 1+\alpha_1 i+\alpha_2 j+\alpha_3 k, \alpha_0,\alpha_1,\alpha_2,\alpha_3 \in \mathbb{R}\}$. Given a quaternion $x=x_0 1+x_1 i+x_2 j+x_3 k$ we call to x_0 the real part of x we denote it by Re (x) and we call to $x_1 i+x_2 j+x_3 k$ the imaginary part of x and we write $\operatorname{Im}(x)=x_1 i+x_2 j+x_3 k$.

One says that a quaternion x is a pure quaternion if Re (x)=0 and if $x=$ Re (x) one says that the quaternion x is a real number.

For discovering the equalities for multiplication describe above we must use the diagram of Fano, see Figure 2.

When we multiply to elements of the set $\{i,j,k\}$ we use the rule: when we multiply two elements in clockwise sense we get the next element, so for instance we have $jk=i$, but if we multiply them in the counterclockwise sense we obtain the next but with minus sign $kj=-i$.

And, therefore we obtain Table 1 of multiplication of two elements of A.

If we consider $x=x_0 1+x_1 i+x_2 j+x_3 k$ and $y=y_0 1+y_1 i+y_2 j+y_3 k$ we obtain the following expression for the product $x*y$,

$$x*y=x_0y_0-(x_1y_1+x_2y_2+x_3y_3)+x_0(y_1 i+y_2 j+y_3 k)+y_0(x_1 i+x_2 j+x_3 k)+(x_2y_3-x_3y_2)i+(x_3y_1-x_1y_3)j+(x_1y_2-x_2y_1)\mathrm{k}.$$

Using the notation $\operatorname{Re}(x)=x_0$ and $\operatorname{Im}(x)=x_1 i+x_2 j+x_3 k$ we conclude that

$$x*y=\operatorname{Re}(x)\operatorname{Re}(y)-\operatorname{Im}(x)|\operatorname{Im}(y)+\operatorname{Re}(x)\operatorname{Im}(y)+\operatorname{Re}(y)\operatorname{Im}(x)+\operatorname{Im}(x)\times\operatorname{Im}(y).$$

Table 1. Table of multiplication of quaternions.

$x * y$	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

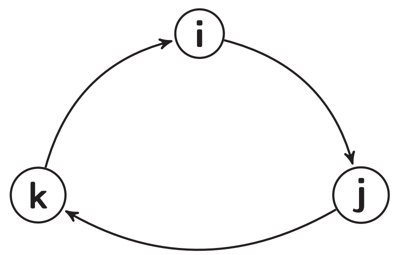

Figure 2. Diagram of Fano for quaternions.

For a quaternion $x = x_0 1 + x_1 i + x_2 j + x_3 k$ we define we define $||x||$ by the the equality $||x|| = \sqrt{x_0^2 + x_1^2 + x_2^2 + x_3^2}$. Now, we will show that $\overline{x * y} = \overline{y} * \overline{x}$. Hence, we have the following calculations:

$$\begin{aligned}\overline{x * y} &= \overline{(\operatorname{Re}(x) + \operatorname{Im}(x)) * (\operatorname{Re}(y) + \operatorname{Im}(y))} \\ &= \overline{\operatorname{Re}(x)\operatorname{Re}(y) - \operatorname{Im}(x) | \operatorname{Im}(y) + \operatorname{Re}(x)\operatorname{Im}(y) + \operatorname{Re}(y)\operatorname{Im}(x) + \operatorname{Im}(x) \times \operatorname{Im}(y)} \\ &= \operatorname{Re}(x)\operatorname{Re}(y) - \operatorname{Re}(x)\operatorname{Im}(y) - \operatorname{Re}(y)\operatorname{Im}(x) - \operatorname{Im}(x) | \operatorname{Im}(y) - \operatorname{Im}(x) \times \operatorname{Im}(y) \\ &= (\operatorname{Re}(y) - \operatorname{Im}(y)) * (\operatorname{Re}(x) - \operatorname{Im}(x)) \\ &= \left(\overline{\operatorname{Re}(y) + \operatorname{Im}(y)}\right) * \left(\overline{\operatorname{Re}(x) + \operatorname{Im}(x)}\right) \\ &= \overline{y} * \overline{x}.\end{aligned}$$

In a similar way, we deduce that $||x|| = \sqrt{x * \overline{x}}$. The inverse of a nonzero quaternion x is define as an element x^{-1} such that $x * x^{-1} = x^{-1} * x = 1$. But since $x * \overline{x} = \overline{x} * x = x | \overline{x} = ||x||^2 = x^{2*}$, where $x^{2^*} = x * x$. then, $x^{-1} = \frac{\overline{x}}{||x||^2}$.

Now, we will introduce the real linear space of octonions. $\mathcal{A} = \{x_0 + x_1 f_1 + x_2 f_2 + x_3 f_3 + x_4 f_4 + x_5 f_5 + x_6 f_6 + x_7 f_7, x_i \in \mathbb{R}, x_0 \in \mathbb{R}, x_i \in \mathbb{R}, i = 1, \cdots, 7\}$ where the elements of the basis $B = \{1, f_1, f_2, f_3, f_4, f_5, f_6, f_7\}$ of A satisfy the rules of multiplication presented in Table 2 below and deduced using the diagram presented in Figure 3.

Now, we fulfill the table recurring to the diagram of Fano, for instance we have $f_2 f_4 = f_6$ but we have $f_4 f_2 = -f_6$ since the sense from f_4 to f_2 is contrary to the sense of line that contains f_4, f_2 and f_6.

The conjugate of the octonion $x = x_0 + x_1 f_1 + x_2 f_2 + x_3 f_3 + x_4 f_4 + x_5 f_5 + x_6 f_6 + x_7 f_7$ is $\overline{x} = x_0 - x_1 f_1 - x_2 f_2 - x_3 f_3 - x_4 f_4 - x_5 f_5 - x_6 f_6 - x_7 f_7$ and the real part of the octonion x is $\operatorname{Re}(x) = x_0$ and the imaginary part of the octonion is $\operatorname{Im}(x) = x_1 f_1 + x_2 f_2 + \cdots + x_7 f_7$. We define

$$||x||| = \sqrt{x * \overline{x}} = \sqrt{x_0^2 + x_1^2 + x_2^2 + \cdots + x_7^2}.$$

Table 2. Table of multiplication of octonions.

$x*y$	1	f_1	f_2	f_3	f_4	f_5	f_6	f_7
1	1	f_1	f_2	f_3	f_4	f_5	f_6	f_7
f_1	f_1	-1	f_3	$-f_2$	f_5	$-f_4$	$-f_7$	f_6
f_2	f_2	$-f_3$	-1	f_1	f_6	f_7	$-f_4$	$-f_5$
f_3	f_3	f_2	$-f_1$	-1	f_7	$-f_6$	f_5	$-f_4$
f_4	f_4	$-f_5$	$-f_6$	$-f_7$	-1	f_1	f_2	f_3
f_5	f_5	f_4	$-f_7$	f_6	$-f_1$	-1	$-f_3$	f_2
f_6	f_6	f_7	f_4	$-f_5$	$-f_2$	f_3	-1	$-f_1$
f_7	f_7	$-f_6$	f_5	f_4	$-f_3$	$-f_2$	f_1	-1

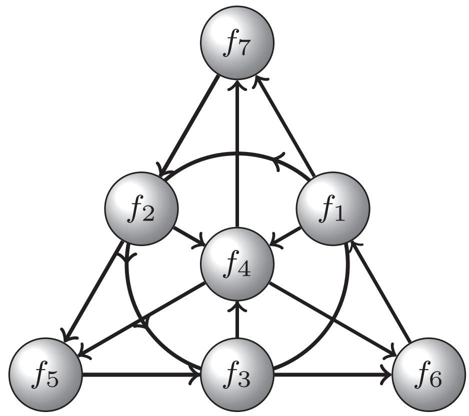

Figure 3. Diagram of Fano for octonions.

Using the table of multiplication 2 we obtain for $x = x_0 + x_1f_1 + x_2f_2 + x_3f_3 + x_4f_4 + x_5f_5 + x_6f_6 + x_7f_7$ and for $y = y_0 + y_1f_1 + y_2f_2 + y_3f_3 + y_4f_4 + y_5f_5 + y_6f_6 + y_7f_7$ we obtain

$$\begin{aligned} x{*}y &= x_0y_0 + x_0(y_1f_1 + y_2f_2 + y_3f_3 + y_4f_4 + y_5f_5 + y_6f_6 + y_7f_7)+ \\ &+ y_0(x_1f_1 + x_2f_2 + x_3f_3 + x_4f_4 + x_5f_5 + x_6f_6 + x_7f_7)+ \\ &- x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4 - x_5f_5 - x_6y_6 - x_7y_7+ \\ &+ (x_2y_3 - x_3y_2)f_1 + (x_3y_1 - x_1y_3)f_2 + \cdots + (x_1y_2 - x_2y_1)f_7. \end{aligned}$$

We must note that: $x * y = \operatorname{Re}(x)\operatorname{Re}(y) + \operatorname{Re}(x)\operatorname{Im}(y) + \operatorname{Re}(y)\operatorname{Im}(x) - \operatorname{Im}(x)|\operatorname{Im}(y) + \operatorname{Im}(x) \times \operatorname{Im}(y)$. Proceeding like we have done for the quaternions we deduce that $\overline{x * y} = \overline{y} * \overline{x}$. We define the norm of an octonion as being $||x|| = \sqrt{x_0^2 + x_1^2 + x_2^2 + \cdots + x_7^2}$. We, must say, again that $||x|| = \sqrt{x{*}\overline{x}}$. Let x be an octonion such that $||x|| \neq 0$ then the inverse of x is $x^{-1} = \frac{\overline{x}}{||x||^2}$, since $x{*}\, x^{-1} = x^{-1} * x = 1$.

Acknowledgments

Luís Vieira was partially supported by CMUP (UID/MAT/00144/2019), which is funded by FCT with national (MCTES) and European structural funds through the programs FEDER, under the partnership agreement PT 2020.

The author are grateful to the anonymous referee for a careful checking of the details and for helpful comments that improved this paper.

References

1. Faybusovich L (1997), Linear systems in Jordan algebras and primal-dual interior-point algorithms. J Comput Appl Math 86, 149–175.
2. Faybusovich L (1997), Euclidean Jordan algebras and Interior-point Algorithms. Positivity 1, 331–357.
3. Faybusovich L (2002), A jordan algebraic approach to potential reduction algorithms. Math Zeitschr 239, 117–129.
4. Faybusovich L, Tsuchiya T (2003), Primal-dual algorithms and infinite-dimensional Jordan algebras of finite rank. Math Program 97, 3, 471–493.
5. Schmieta SH, Alizadeh F (2001), Associative and Jordan algebras and polynomial time Interior point alghoritms for symmetric cones. Math Oper Res 26, 3, 543–564.
6. Schmieta SH, Alizadeh F (2003), Extension of commutative class of primal dual interior point algorithms to symmetric cones. Math Program 96, 409–438.
7. Alizadeh F, Goldfarb D (2003), Second order cone programming. Math Program 95, 1, 3–51.
8. Cardoso DM, Vieira LA (2006), On the optimal parameter of a self-concordant barrier over a symmetric cone. Eur J Oper Res 169, 1148–11157.
9. Chang YL, Yang CY (2014), Some useful inequalities via trace function method in Euclidean Jordan algebras. Numer Algebra Control Optim 4, 1, 39–48.
10. Monteiro RDC, Tsuchiya T (2000), Polynomial convergence of primal-dual algorithms for the second-order cone programs based on the MZ-family of directions. Math Program Ser A 88, 61–83.
11. Vieira LA, Mano VM (2015), Generalized Krein parameters of a strongly regular graph. Appl Math 6, 37–45.
12. Mano VM, Martins EA, Vieira LA (2013), On generalized binomial series and strongly regular graphs. Proyecciones J Math 4, 393–408.

13. Mano VM, Vieira LA (2011), Admissibility conditions and asymptotic behavior of strongly regular graphs. Int J Math Models Methods Appl Sci Methods 6, 1027–1033.
14. Mano VM, Vieira LA (2014), Alternating Schur series and necessary conditions for the existence of strongly regular graphs. Int J Math Models Methods Appl Sci Methods 8, 256–261.
15. Letac G, Wesolowski J (2011), Why Jordan algebras are natural in statistics:quadratic regression implies whishart distribuitions. Bull Soc Math France 139, 129–144.
16. Massan H, Neher E (1998), Estimation and testing for lattice condicional independence models on Euclidean Jordan algebras. Ann Stat 1, 1051–1081.
17. Malley JD (1994), Statistical applications of Jordan algebras, Springer-Verlag, Berlin.
18. Gowda MS, Tao J (2011), Some inequalities involving determinants, eigenvalues, and Schur complements in Euclidean Jordan algebras. Positivity 15, 381–399.
19. Gowda MS, Tao J (2011), The Cauchy interlacing theorem in simple Euclidean Jordan algebras and some consequences. Linear Multilinear Algebra 59, 1563–5139.
20. Sznadjer R, Gowda MS, Moldovan MM (2012), More results on Schur complements in Euclidean Jordan algebras. J Global Optim 53, 121–134.
21. Moldovan MM, Gowda MS (2009), Strict diagonal dominance and a Gersgorin type theorem in Euclidean Jordan algebras. Linear Algebra Appl 431, 148–161.
22. Gowda MS, Tao J (2011), Some inequalities involving determinants, eigenvalues, and Schur complements in Euclidean Jordan algebras. Positivity 15, 381–399.
23. Lim Y, Kim J, Faybusovich L (2003), Simultaneous diagobalization on simple Euclidean Jordan algebras and its applications. Forum Math 15, 4, 639.
24. Tao J, Kong L, Luo Z, Xiu N (2014), Some majorization inequalities in Euclidean jordan algebras. Linear Algebra Appl 461, 92–122.
25. McCrimmon K (2003), A taste of Jordan algebras, Springer-Verlag, Berlin.
26. Jacobson N (2008), Structure and representations of Jordan algebras, American Mathematical Society, Providence.
27. Baes M (2006), Spectral functions and smoothing techniques on Jordan Algebras, PhD thesis, Universitaté Catholic de Louvain.
28. Faraut J, Korányi A (1994), Analysis on symmetric cones, Oxford Science Publications, Oxford.
29. Vieira MVC (2007), Jordan algebraic approach to symmetric optimization, PhD thesis, University Nova de Lisboa.
30. Moldovan MM (2009), A Gershgorin type theorem, special inequalities and simultaneous stability in Euclidean jordan algebras, PhD thesis, University of Maryland, Baltimore County.
31. Alizadeh F, Schmieta SH (2000), Symmetric cones, potential reduction methods and word-by-word extensions, in: R Saigal, L Vandenberghe, H Wolkowicz (Eds.), Handbook of semidefinite programming, Theory, Algorithms and Applications, Kluwer Academic Publishers, Massachusetts, pp. 195–259.
32. Alizadeh F (2012), An introduction to formally real Jordan algebras and their applications in optimization, in: MF Anjos, JB Lasserre (Eds.), The handbook on semidefinite, conic and polynomial optimization, Springer, Boston, pp. 229–337.
33. Bose RC (1963), Strongly regular graphs, partial geometries and partially balanced designs. Pacific J Math 13, 384–419.
34. Brouwer AE, Haemers WH (2010), Spectra of graphs, Springer, Amsterdam.
35. Lint JHV, Wilson RM (1992), A course in combinatorics, University Press, Cambridge.
36. Scott LL (1973), A condition on Higman's parameters. Notices Am Math Soc 20, A-7.
37. Ghatei D (2003), The octonions, PhD thesis, School of Mathematics of The University of Birmingham, Birmingham.
38. Conway JH, Smith DA (2003), On quaternions and octonions, Their geometry, arithmetic and symmetry, CRC Press, New York.

Cite this article as: Vieira L 2019. Euclidean Jordan algebras and some conditions over the spectraof a strongly regular graph. 4open, **2**, 21.