4open 2023, **6**, 2
© V.A. González-López et al., Published by EDP Sciences, 2023
https://doi.org/10.1051/fopen/2023001

4open

Available online at:
www.4open-sciences.org

Special Issue Statistical Inference in Markov Processes and Copula Models

RESEARCH ARTICLE OPEN ACCESS

A method for the elicitation of copulas

Verónica Andrea González-López and Vinícius Litvinoff Justus*

Department of Statistics, University of Campinas, Sergio Buarque de Holanda, 651, Campinas, S.P. CEP: 13083-859, Brazil

Received 15 September 2022, Accepted 4 January 2023

Abstract – In this paper, we introduce a method to construct copulas. The method is based on combining the partial derivatives of two copulas. We prove that the proposed method provides a copula. Then, we exemplify the application of the method in several cases, illustrating the versatility of the method. We also prove that using copulas from the family introduced in Rodríguez-Lallena and Úbeda-Flores (2004) [Stat Probab Lett 66, 3, 315–325. https://doi.org/10.1016/j.spl.2003.09.010], the method provides a copula inside that family.

Keywords: Dependence, Exchangeability, Representations of de Finetti

Introduction

From the fact that it is convenient to have a variety of copulas at our disposal, amplifying the options for modeling, in this paper we propose a method to build copulas. The method provides a tool to elicitate a copula combining other copulas. We limit the exposition to dimension 2, using 2-copulas to build 2-copulas, but the technique can be easily extended to d-copulas, with $d > 2$.

Consider U and V two absolutely continuous random variables with Uniform distributions in [0, 1], our goal is to build a 2-copula between U and V. For to do that we consider Θ, a third absolutely continuous random variable linking U and V. The proposal of this paper requires a previous knowledge of the existing dependence between (a) U and Θ, and (b) V and Θ. To simplify the exposition and because we want to propose a method for elicitating 2-copulas between U and V, we assume also that Θ is absolutely continuous random variable with Uniform distribution in [0, 1]. In this way, for the construction of a copula between U and V, it is necessary to postulate a 2-copula between U and Θ and another 2-copula between V and Θ.

If we want to give a clear meaning to Θ, which rudimentarily speaking is a latent variable embedded in the process of dependence of the variables U and V, we appeal to the principles of de Finetti's representation theorems. Those theorems, proved in de Finetti [1] and Hewitt and Savage [2], indicate that under certain conditions, see Aldous [3] and Mai [4], there is a random variable Θ allowing the conditional independence between U and V. Then, those results provide the necessary intuition behind the method proportioned in this paper.

For practicality and to expand the scope of the method, in this paper, we will not necessarily refer to Θ as the variable identified in de Finetti [1] and Hewitt and Savage [2], since we want to present the idea that it is enough to have two 2-copulas, between U and Θ and V and Θ to postulate the resulting 2-copula between U and V.

The content of this paper is the following, Copula construction method section shows the method for the construction of 2-copulas, we also expose examples showing different situations. Rodríguez-Lallena and Úbeda-Flores Family section shows the performance of the method on the family of 2-copulas proposed in Rodríguez-Lallena and Úbeda-Flores [5]. Conclusion section shows the conclusions.

A copula construction method

In this section, we present the notion of copula and some implications of its definition, on partial derivatives. These notions allow us to define the copula elicitation method that is based on a theorem that we postulate and prove in this section. We exemplify the applicability of the method by addressing examples that allow us to assess the scope of the method.

*Corresponding author: v245362@dac.unicamp.br; viniliff@gmail.com

Definition 2.1

A 2-copula is a function C: $[0, 1]^2 \to [0, 1]$ *with the following properties,*

- $\forall u, v \in [0,1]$, $C(u,0) = C(0,v) = 0$, and $C(u,1) = u$, $C(1,v) = v$;
- for every $u_i, v_i \in [0,1]$, $i = 1,2$, such that $u_1 \le u_2$ and $v_1 \le v_2$,

$$V_C([u_1,u_2] \times [v_1,v_2]) = C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \ge 0.$$

Definition 2.1 implies that a 2-copula is a joint cumulative distribution of variables with Unif([0, 1]) marginal distributions. In this paper, we simply use the term copula (and not 2-copula) since the technique and concepts can be extended to dimensions greater than 2, and we limit ourselves to their development in dimension 2.

From Theorem 2.2.7 of Nelsen [6], if C is a copula, for any $v \in [0, 1]$, the partial derivative $\frac{\partial C(u,v)}{\partial u}$ exists for almost all u (in the sense of Lebesgue measure). Also the function $v \to \frac{\partial C(u,v)}{\partial u}$ is nondecreasing almost everywhere on [0, 1]. In similar way this properties are valid for the coordinate v. We see below an example.

Example 2.2

Let C *be a function* $C : [0,1]^2 \to [0,1]$ *given by* $C(u,v) = \frac{uv}{u+v-uv}$, $u,v \in [0,1]$. *Then,* C *is a copula. Note that this copula is coming from the Ali–Mikhail–Haq family of copulas, with parameter equal to 1, since its general expression is* $\frac{uv}{1-\theta(1-u)(1-v)}$, $\theta \in [-1,1)$. *And,*

$$\frac{\partial C(u,v)}{\partial u} = \frac{v^2}{(u+v-uv)^2},\ u,v \in [0,1].$$

Functions like $v \to \frac{\partial C(u,v)}{\partial u}$ are frequently used, together with the Sklar's theorem, for generating samples from joint distributions, prescribed the marginal distributions and the copula, see for instance Nelsen [6]. In a closer view, note that

$$\frac{\partial C(u,v)}{\partial u} = \lim_{\delta \to 0} \frac{C(u+\delta,v) - C(u,v)}{\delta} = \text{Prob}\,(V \le v | U = u). \tag{1}$$

In a similar way as given by equation (1), we have that $\frac{\partial C(u,v)}{\partial v} = Prob\,(U \le u | V = v)$. We show in Theorem 2.3 how the functions $\frac{\partial C(u,v)}{\partial u}$ and $\frac{\partial C(u,v)}{\partial v}$ can be used to propose a copula. More specifically, Theorem 2.3 constitutes the basis of a new method for elicitation of copulas. We consider two copulas C_1 and C_2, following Definition 2.1, we calculate their partial derivatives (Eq. (1)) and finally we combine them (this is the proposal of the theorem) obtaining a new copula.

Theorem 2.3

If C_1 *and* C_2 *are two copulas, see Definition 2.1,*

$$C(u,v) = \int_0^1 \frac{\partial C_1(u,t)}{\partial t} \frac{\partial C_2(v,t)}{\partial t} dt, \tag{2}$$

is a copula.

Proof. Property *i.* of Definition 2.1: from equation (1) applied to C_1,

$$C(u,v) = \int_0^1 Prob\,(U \le u | \Theta = t) \times \frac{\partial C_2(v,t)}{\partial t} dt. \tag{3}$$

If $u = 0$, Prob $(U \le 0 | \Theta = t) = 0$, then $C(0,\ v) = 0$, $\forall v \in [0,\ 1]$, since $\frac{\partial C_2(v,t)}{\partial t}$ is well defined ($\frac{\partial C_2(v,t)}{\partial t} \in [0,1]$, by Theorem 2.2.7 in Nelsen [6]). Similarly, $C(u,\ 0) = 0$, $\forall u \in [0,\ 1]$.

If $u = 1$, Prob $(U \le 1 | \Theta = t) = 1$, then, equation (3) is $C(1,v) = \int_0^1 \frac{\partial C_2(v,t)}{\partial t} dt = C_2(v,1) = v$ since the marginal distribution of the copula C_2 is Uniform in [0, 1]. Similarly, $C(u,\ 1) = u$, $\forall u \in [0,\ 1]$.

Property *ii.* of Definition 2.1: consider u_i, $v_i \in [0, 1]$, $i = 1, 2$, such that $u_1 \le u_2$ and $v_1 \le v_2$, we need to prove $V_C(A) \ge 0$, for $A = [u_1,\ u_2] \times [v_1,\ v_2]$. Write C as

$$C(u,v) = \int_0^1 \text{Prob}\,(U \le u|\Theta = t) \times \text{Prob}\,(V \le v|\Theta = t)\text{d}t,$$

then,

$$\begin{aligned} V_C(A) &= \int_0^1 [\text{Prob}\,(U \le u_2|\Theta = t) - \text{Prob}\,(U \le u_1|\Theta = t)][\text{Prob}\,(V \le v_2|\Theta = t) - \text{Prob}\,(V \le v_1|\Theta = t)]\text{d}t \\ &= \int_0^1 \text{Prob}\,(u_1 < U \le u_2|\Theta = t) \times \text{Prob}\,(v_1 < V \le v_2|\Theta = t)\text{d}t \ge 0, \end{aligned} \tag{4}$$

where equation (4) follows from the assumptions $u_1 \le u_2$ and $v_1 \le v_2$.

Then, C is a function following Definition 2.1. □

Thus, the strategy that we propose in this article seeks to fabricate a copula between U and V, with the help of a third quantity, say Θ.

We can then say that via knowledge of two copulas C_1 and C_2, Theorem 2.3 states how to combine such copulas to build a copula between U and V. This is, if C_1 is a copula between U and Θ and C_2 is a copula between V and Θ, we calculate $\frac{\partial C_1(u,t)}{\partial t}$ and $\frac{\partial C_2(v,t)}{\partial t}$, then, through equation (2) we obtain a copula between U and V.

Equation (2) shows that the role that the variable Θ plays is to make the variables U and V conditionally independent given $\Theta = t$, since,

$$\int_0^1 \frac{\partial C_1(u,t)}{\partial t}\frac{\partial C_2(v,t)}{\partial t}\text{d}t = \int_0^1 \text{Prob}\,(U \le u|\Theta = t) \times \text{Prob}\,(V \le v|\Theta = t)\text{d}t.$$

The variable Θ that allows U and V to be linked could be interpreted as being the variable identified by de Finetti representations, or a transformation of it, since in Theorem 2.3, Θ has Uniform distribution on [0, 1]. Although this aspect escapes the focus of this article, we recommend the reader some papers on the subject, de Finetti [1], Hewitt and Savage [2].

We present three examples, which seek to show the potential of equation (2), as a method of constructing copulas. The first produces a copula that generalizes the Farlie–Gumbel–Morgenstern family (see Eyraud [7]), the second produces a copula in the Ali–Mikhail–Haq family (see Ali et al. [8]), the third example shows that although we choose two copulas C_1 and C_2 in the same family of copulas, equation (2) does not necessarily generate a copula in the same family of C_1 and C_2.

The following example shows a case that combines two copulas, one of them being a member of the Farlie-Gumbel–Morgenstern family. Through the method proposed by Theorem 2.3, we build a copula that is a generalization of the Farlie–Gumbel–Morgenstern copula.

Example 2.4

Consider two copulas, C_1 *and* C_2, $C_1(u,t) = u - u(1-t)^{\frac{1}{u}}$ *and* $C_2(v,t) = vt + \alpha v(1-v)t(1-t)$, *with* $\alpha \in [-1,1], u, v, t \in [0,1]$,

$$\frac{\partial C_1(u,t)}{\partial t} = (1-t)^{\frac{1-u}{u}}, \frac{\partial C_2(v,t)}{\partial t} = v + \alpha v(1-v)(1-2t).$$

In this case, equation (2) *is*

$$C(u,v) = uv\left\{1 + \alpha\frac{(u-1)}{(u+1)}(v-1)\right\}, \quad u,v \in [0,1],$$

which is a type of generalization of the Farlie–Gumbel–Morgenstern family of copulas (see Rodríguez-Lallena and Úbeda-Flores [5]), and note that C_2 *is the Farlie–Gumbel–Morgenstern copula, with parameter* α.

The following example shows a case that produces a copula, member of the Ali–Mikhail–Haq family, see Ali et al. [8] and Nelsen [6].

Example 2.5

Consider two copulas, C_1 *and* C_2, $C_1(u,t) = u - u(1-t)^{\frac{1}{u}}, C_2(v,t) = v - v(1-t)^{\frac{1}{v}}, u, v, t \in [0,1]$,

$$\frac{\partial C_1(u,t)}{\partial t} = (1-t)^{\frac{1-u}{u}}, \frac{\partial C_2(v,t)}{\partial t} = (1-t)^{\frac{1-v}{v}}.$$

Now, in this case, equation (2) is

$$C(u,v) = \frac{uv}{u+v-uv}, \quad u,v \in [0,1],$$

which is a case of the Ali–Mikhail–Haq copula (with parameter equal to 1). Note that this copula is the one used in Example 2.2.

The following example exposes a case that shows that even if the copulas C_1 and C_2 are coming from the same copula's family, the resulting copula (by the application of Theorem 2.3) is not a member of the same family of C_1 and C_2.

Example 2.6

Let C_1 and C_2 be two Gumbel–Hougaard copulas (see Hougaard [9]), $C_1(u,t) = ut\exp(-\alpha\ln(u)\ln(t))$, $C_2(v,t) = vt\exp(-\beta\ln(v)\ln(t))$, u, v, $t \in [0,1]$, with α, $\beta \in (0,1]$,

$$\frac{\partial C_1(u,t)}{\partial t} = u(1-\alpha\ln(u))t^{-\alpha\ln(u)}, \frac{\partial C_2(v,t)}{\partial t} = v(1-\beta\ln(v))t^{-\beta\ln(v)}. \qquad \square$$

Equation (2) is

$$C(u,v) = \frac{uv[1-\alpha\ln(u)][1-\beta\ln(v)]}{1-\alpha\ln(u)-\beta\ln(v)}, \quad u,v \in [0,1].$$

That is, the constructed copula is not a member of Gumbel–Hougaard family.

In this section, we introduce the notion of partial derivative $\frac{\partial C(s,t)}{\partial t}$, fundamental for the copula construction method, which is the result of Theorem 2.3. This notion is familiar in random vector simulation methods, as described in Nelsen [6]. The introduced method proposes to construct a copula between two variables U and V, using a third variable Θ and combining the copula between U and Θ and the copula between V and Θ, through equation (2). Θ represents a random variable connecting U and V. We conclude the section by presenting three examples illustrating the construction of three different copulas.

The next subsection deals with the case of the family of copulas introduced in Rodríguez-Lallena and Úbeda-Flores [5]. Our goal is to identify if using two copulas C_1 and C_2 from that family, Theorem 2.3 gives us a copula in that family. We know that it is not always the case, see for instance Example 2.6. But Example 2.4 seems to indicate that in certain families (like Farlie–Gumbel–Morgenstern family) the construction of copulas proposed by Theorem 2.3 could behave in a closed way.

Rodríguez–Lallena and Úbeda–Flores Family

We start this section by introducing the analytic form of a family of copulas present in Rodríguez-Lallena and Úbeda-Flores [5]. We formalize the conditions for such a function to be a copula. It should be noted that the analytic form of this copula involves a huge range of particular cases, widely investigated in the literature. For instance it includes families of copulas with quadratic sections, see Quesada Molina and Rodríguez-Lallena [10], the family of copulas with cubic cross-sections proposed in Nelsen et al. [11], and families of positive quadrant dependent copulas as the one introduced in Lai and Xie [12]. See also Amblard and Girard [13] to obtain details about several properties of semi parametric families of copulas in this class.

Consider the function $C : [0,1]^2 \to \mathbb{R}$,

$$C(u,v) = uv + f(u)g(u), f,g\text{ non}-\text{zero real functions}, \quad f,g : [0,1] \to \mathbb{R}. \tag{5}$$

Following Theorem 2.3 of Rodríguez-Lallena and Úbeda-Flores [5], equation (5) is a copula, if and only if, f and g are absolutely continuous, $f(0) = f(1) = g(0) = g(1) = 0$, and under the existence of the derivatives of f and g, say, on A_f and A_g, respectively,

$$\alpha_f = \inf\{f'(u), u \in A_f\} < 0, \quad \beta_f = \sup\{f'(u), u \in A_f\} > 0, \tag{6}$$

$$\alpha_g = \inf\{g'(u), u \in A_g\} < 0, \quad \beta_g = \sup\{g'(u), u \in A_g\} > 0, \tag{7}$$

and, the next condition is true,

$$-1 \le \min\{\alpha_f\beta_g, \beta_f\alpha_g\}. \tag{8}$$

The family of copulas that responds to the form given by equation (5) generalizes the Farlie–Gumbel–Morgenstern copula and others that were developed, based on the idea of constructing copulas that are disturbances of the independence copula, $\Pi(u, v) = uv$, reaching for example weak dependence types. It is worth mentioning here that the independence copula $\Pi(u, v)$ responds to Boltzmann–Gibbs–Shannon's notion of entropy, while the copula given by equation (5) responds to Tsallis–Havrda–Chavát's entropy (see García et al. [14]). That is, while the copula that maximizes the entropy of Boltzmann–Gibbs–Shannon is the copula Π, the copula that maximizes the entropy of Tsallis–Havrda–Chavát is the one given by equation (5). Then, in summary, this last family adopts a practical sense in terms of the quantification of chaos made by the notion of entropy of Tsallis–Havrda–Chavát.

The following corollary shows that the family of copulas introduced in Rodríguez-Lallena and Úbeda-Flores [5] is preserved, by applying the copula construction given by Theorem 2.3. The following result is established using Corollary 2.4 (Rodríguez-Lallena and Úbeda-Flores [5]). Corollary 2.4 adapts what is established by Theorem 2.3 (Rodríguez-Lallena and Úbeda-Flores [5]). Note that Theorem 2.3 shows that under the constraints of equations (6)–(8) the functional form $uv + f(u)g(v)$ is a copula, while Corollary 2.4 investigates a functional form that introduces a constant (a parameter δ), being the investigated form $uv + \delta f(u)g(v)$. The result identifies the range of possible values for such a parameter δ, to guarantee that the equation follows Definition 2.1.

Corollary 3.1

Let C_1 and C_2 be two copulas given by equation (5), $C_1(u,t) = ut + f_1(u)g_1(t)$, $C_2(v, t) = vt + f_2(v)g_2(t)$, then, the copula resulting from equation (2) follows the form given by equation (5),

$$C(u, v) = uv + \delta f_1(u)f_2(v)$$

with $\delta = \int_0^1 g_1'(t)g_2'(t)\,dt$ *and* $\frac{-1}{max\{\alpha_{f_1}\alpha_{f_2}, \beta_{f_1}\beta_{f_2}\}} \le \delta \le \frac{-1}{min\{\alpha_{f_1}\beta_{f_2}, \beta_{f_1}\alpha_{f_2}\}}$, *where* α_{f_i} *and* $\beta_{f_i}, i = 1,2$ *follows* (6).

Proof. $C(u, v) = uv + \delta f_1(u)f_2(v)$ is a direct consequence of equation (2). Already, condition $\frac{-1}{\max\{\alpha_{f_1}\alpha_{f_2}, \beta_{f_1}\beta_{f_2}\}} \le \delta \le \frac{-1}{\min\{\alpha_{f_1}\beta_{f_2}, \beta_{f_1}\alpha_{f_2}\}}$ follows from Corollary 2.4 of Rodríguez-Lallena and Úbeda-Flores [5]. □

What we see in Corollary 3.1 allows us to say that all generalizations of the Farlie–Gumbel–Morgenstern copula, which are particular cases of equation (5), are preserved by the copula construction method proposed in this paper.

Example 3.2

Consider two copulas, C_1 and C_2, $C_1(u,t) = ut + \lambda_1 u^{a_1}(1-u)t^{b_1}(1-t)$ and $C_2(v,t) = vt + \lambda_2 v^{a_2}(1-v)t^{b_2}(1-t)$ with $u, v, t \in [0, 1]$, $a_i, b_i \ge 1$, and $\frac{-1}{max\{A_iG_i, B_iD_i\}} \le \lambda_i \le \frac{-1}{min\{A_iD_i, B_iG_i\}}$, *with* $A_i = -(\frac{a_i}{a_i+1})^{a_i-1}(1+\frac{1}{a_i})^{a_i-1}$, $B_i = \left(\frac{a_i}{a_i+1}\right)^{a_i-1}\left(1-\frac{1}{a_i}\right)^{a_i-1}$, $G_i = -\left(\frac{b_i}{b_i+1}\right)^{b_i-1}\left(1+\frac{1}{b_i}\right)^{b_i-1}$, $D_i = \left(\frac{b_i}{b_i+1}\right)^{b_i-1}\left(1-\frac{1}{b_i}\right)^{b_i-1}$, $i = 1,2$. *For details on the properties of the copulas C_1 and C_2, see specific literature Lai and Xie* [12] *and Rodríguez-Lallena and Úbeda-Flores* [5]. *We can also see the use of these models in practice in* Fernández *et al.* [15].

Define $f_1(u) = \lambda_1 u^{a_1}(1-u)$, $f_2(v) = \lambda_2 v^{a_2}(1-v)$, $g_i(t) = t^{b_i}(1-t)$, $i = 1, 2$. Applying the Corollary 3.1,

$$\delta = \frac{2b_1b_2}{(b_1+b_2+1)(b_1+b_2)(b_1+b_2-1)}$$

and

$$C(u, v) = uv + \lambda_1\lambda_2\delta u^{a_1}(1-u)v^{a_2}(1-v).$$

If we fix the values $a_i = b_i = 1$, $i = 1, 2$, C_1 and C_2 are the usual Farlie–Gumbel–Morgenstern copulas with parameters λ_1 and λ_2 respectively and $-1 \le \lambda_i \le 1$, $i = 1, 2$. Then, $\delta = \frac{1}{3}$ and the resulting copula is also a Farlie–Gumbel–Morgenstern copula with parameter $-\frac{1}{3} \le \lambda_1\lambda_2\,\delta \le \frac{1}{3}$.

The following corollary tells us how the method behaves when only one of the copulas chosen for the construction of C follows equation (5).

Corollary 3.3

Let C_1 and C_2 be two copulas, with C_1 given by equation (5), $C_1(u, t) = ut + f_1(u)g_1(t)$, then, the copula resulting from equation (2) follows the form given by equation (5),

$$C(u,v) = uv + f_1(u)f_2(v) \quad with \quad f_2(v) = \int_0^1 g_1'(t)\frac{\partial C_2(v,t)}{\partial t}dt.$$

Proof. Since $C_1(u, t) = ut + f_1(u)g_1(t)$, as C_1 is a copula, are applied the conditions of Theorem 2.3 of *Rodríguez-Lallena and Úbeda-Flores* [5] (Eqs. (6), (7) and (8)), and $\frac{\partial C_1(u,t)}{\partial t} = u + f_1(u)g_1'(t)$. Then,

$$\frac{\partial C_1(u,t)}{\partial t}\frac{\partial C_2(v,t)}{\partial t} = u\frac{\partial C_2(v,t)}{\partial t} + f_1(u)g_1'(t)\frac{\partial C_2(v,t)}{\partial t}.$$

The form (5) results from C_2 being a copula ($C_2(v, 1) = v$ and $C_2(v, 0) = 0$) and using the Theorem 2.3. □

Example 3.4

Consider two copulas, C_1 and C_2, $C_1(u, t) = ut + \lambda u(1 - u)t(1 - t)$ with $\lambda \in [-1, 1]$. and $C_2(v, t) = (v^{-1} + t^{-1} - 1)^{-1}$, $u, v, t \in [0, 1]$. C_1 is a Farlie–Gumbel–Morgenstern copula and C_2 is a copula coming from the Clayton family with parameter one, see Clayton [16].

Define $f_1(u) = \lambda u(1 - u)$ and $g_1(t) = t(1 - t)$. Applying Corollary 3.3, and since

$$f_2(v) = \int_0^1 g_1'(t)\frac{\partial C_2(v,t)}{\partial t}\mathrm{d}t = \int_0^1 (1-2t)\frac{v^2}{(t+v-tv)^2}\mathrm{d}t = \frac{v(1+2v\ln(v)-v^2)}{(v-1)^2},$$

the copula is $C(u,v) = uv + \lambda u(1-u)\frac{v(1+2v\,ln(v)-v^2)}{(v-1)^2}$, that is, a case of equation (5).

In this section, we investigate how the copula build by equation (2) turn out if we use at least one copula from the copula family introduced in Rodríguez-Lallena and Úbeda-Flores [5]. We present our results in two corollaries. The first of them identifies the analytic form of the resulting copula if both copulas C_1 and C_2 belong to the family introduced in Rodríguez-Lallena and Úbeda-Flores [5]. Corollary 3.1 shows that the resulting copula is also a member of the family given in Rodríguez-Lallena and Úbeda-Flores [5]. Corollary 3.3 gives the analytic form of the resulting copula if only one copula, say C_1, is a member of the family introduced in Rodríguez-Lallena and Úbeda-Flores [5]. And again, the resulting copula is a member of the family introduced in Rodríguez-Lallena and Úbeda-Flores [5]. Both results are exemplified, see Examples 3.2 and 3.4.

Conclusion

In this paper we show how to use the functions $\frac{\partial C_1(u,t)}{\partial t}$ and $\frac{\partial C_2(v,t)}{\partial t}$ to obtain a copula between U and V, where u is a value of U and v a value of V. We prove, in Theorem 2.3 that is only necessary C_1 and C_2 to be copulas to obtain a copula of (U, V) by equation (2). Note that the quantities $\frac{\partial C_1(u,t)}{\partial t}$ and $\frac{\partial C_2(v,t)}{\partial t}$ are also used to generate random vectors with prescribed dependence, and with prescribed marginal distributions, as Nelsen [6] shows. This means that all those functions usually employed to generate vectors can also be used to create copula models. This fact shows the potential of the method proposed in this paper. For a variety of models see Joe [17] and Nelsen [6].

We illustrate the method in three situations, (i) generating a copula that generalizes the Farlie–Gumbel–Morgenstern copula family, see Example 2.4 (Eyraud [7]), (ii) generating an Ali-Mikhail-Haq type copula, see Example 2.5 (Ali et al. [8]), and (iii) showing the type of dependence that results when combining two copulas coming from the Gumbel-Hougaard family, see Example 2.6 (Hougaard [9]).

Returning to the problem cited in the introduction of this paper, we can then postulate an inferential strategy so that from the dependence relationship between U and Θ represented by a copula C_1 and between V and Θ, identified by the copula C_2, it is possible to postulate a dependence relationship between U and V that is generated from equation (2). This type of construction of the dependence between U and V requires the identification of an auxiliary variable Θ, linking U and V. For a subjective reading/interpretation of the meaning of the variable Θ see de Finetti [1], Hewitt and Savage [2] and O' Neill [18].

We finish our contribution with two results (Corollaries 3.1 and 3.3) that identify the behavior of our method when applied to copulas from the family introduced in Rodríguez-Lallena and Úbeda-Flores [5]. We see that the method is able to preserve it, guaranteeing that the copula between U and V is a member of the Rodríguez-Lallena and Úbeda-Flores [5] family. Corollaries 3.1 and 3.3 are exemplified in popular cases, see Examples 3.2 and 3.4 respectively.

Conflict of interest

Authors declared no conflict of interests.

Acknowledgments

Vinícius Litvinoff Justus gratefully acknowledges the support, provided by CNPq with a fellowship of scientific initiation from University of Campinas. The authors wish to thank the referees and editors for their many helpful comments and suggestions on an earlier draft of this paper.

References

1. de Finetti B (1929), Funzione caratteristica di un fenomeno aleatorio, in Atti del Congresso Inter nazionale dei Matematici: Bologna del 3 al 10 de settembre di 1928, Vol. 6 (Comunicazioni, sezione IV (A)-V-VII), pp. 179–190.
2. Hewitt E, Savage LJ (1955), Symmetric measures on Cartesian products. Trans Am Math Soc 80, 2, 470–501. https://doi.org/10.2307/1992999.
3. Aldous DJ (1985), Exchangeability and related topics, in: P.L. Hennequin (Ed.), École d'Été de Probabilités de Saint-Flour XIII – 1983, Vol. 1117, Lecture Notes in Mathematics, Springer, Berlin, Heidelberg. https://doi.org/10.1007/BFb0099421.
4. Mai J-F (2020), The infinite extendibility problem for exchangeable real-valued random vectors. Probab Surv 17, 677–753. https://doi.org/10.1214/19-PS336.
5. Rodríguez-Lallena JA, Úbeda-Flores M (2004), A new class of bivariate copulas. Stat Probab Lett 66, 3, 315–325. https://doi.org/10.1016/j.spl.2003.09.010.
6. Nelsen RB (2006), An introduction to copulas, Springer-Verlag, New York. https://doi.org/10.1007/0-387-28678-0.
7. Eyraud H (1938), Les principes de la mesure des correlations. Ann Univ Lyon Series A 1, 30–47.
8. Ali MM, Mikhail NN, Haq MS (1978), A class of bivariate distributions including the bivariate logistic. J Multivar Anal 8, 3, 405–412. https://doi.org/10.1016/0047-259X(78)90063-5.
9. Hougaard P (1986), A class of multivariate failure time distributions. Biometrika 73, 3, 671–678. https://doi.org/10.1093/biomet/73.3.671.
10. Quesada-Molina JJ, Rodríguez-Lallena JA (1995), Bivariate copulas with quadratic sections. J Nonparametr Statist 5, 4, 323–337. https://doi.org/10.1080/10485259508832652.
11. Nelsen RB, Quesada-Molina JJ, Rodríguez-Lallena JA (1997), Bivariate copulas with cubic sections. J Nonparametr Statist 7, 3, 205–220. https://doi.org/10.1080/10485259708832700.
12. Lai CD, Xie M (2000), A new family of positive quadrant dependent bivariate distributions. Stat Probab Lett 46, 4, 359–364. https://doi.org/10.1016/S0167-7152(99)00122-4.
13. Amblard C, Girard S (2002), Symmetry and dependence properties within a semiparametric family of bivariate copulas. J Nonparametr Statist 14, 6, 715–727. https://doi.org/10.1080/10485250215322.
14. García JE, González-López VA, Nelsen RB (2016), The structure of the class of maximum Tsallis–Havrda–Chavát Entropy Copulas. Entropy 18, 7, 264. https://doi.org/10.3390/e18070264.
15. Fernández M, González-López VA, Rifo LR (2015), A note on conjugate distributions for copulas. Math Methods Appl Sci 38, 18, 4797–4803. https://doi.org/10.1002/mma.3394.
16. Clayton DG (1978), A model for association in bivariate life tables and its application in epidemiological studies of familial tendency in chronic disease incidence. Biometrika 65, 1, 141–151. https://doi.org/10.1093/biomet/65.1.141.
17. Joe H (2014), Dependence modeling with copulas. Chapman and Hall/CRC, New York. https://doi.org/10.1201/b17116.
18. O'Neill B (2009), Exchangeability, correlation, and bayes' effect. Int Stat Rev 77, 2, 241–250. https://doi.org/10.1111/j.1751-5823.2008.00059.x.

Cite this article as: González-López VA & Litvinoff Justus V. 2023. A method for the elicitation of copulas. 4open, **6**, 2.